

POLYNOMIAL INVARIANTS FOR LINKS AND TIED  
LINKS  
(UNDER CONSTRUCTION, 2020)

JESÚS JUYUMAYA

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## 1. BACKGROUND

**Definition 1.** A knot is a subset of  $\mathbb{R}^3$  homeomorphic to  $S^1$ . A link is a disjoint union of  $n$  knots,  $n$  is called the number of components of the link.

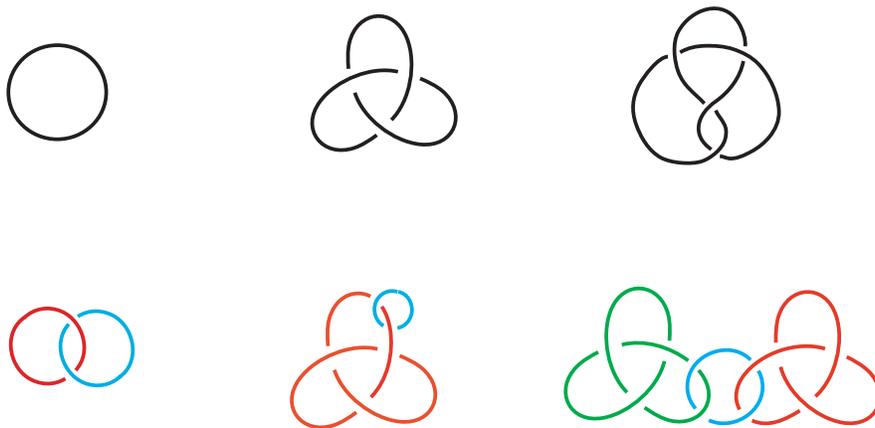


FIGURE 1

## KnotsLinks

In Fig. 1: the first knot is called the unknot, the second is the left trefoil knot, the third is the figure-eight knot, the fourth is the Hopf link, the fifth is a link with two components and the sixth is a link with three components.

Since a knot is a simple closed curve we can provide it with an orientation, in such case we say that the *knot is oriented*. If each component of a link is oriented, we say that *the link is oriented*.



FIGURE 2

## KnotsLinksOriented

**Definition 2.** Two links  $L_1$  and  $L_2$  (both oriented or not) are called *ambient isotopic* (or *equivalent*) if there exists an ambient isotopy of  $\mathbb{R}^3$  that carries  $L_1$  in  $L_2$ , that is, there is a continue map  $\phi : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$  such that:

- (1)  $\phi(L_1, 0) = L_1$  and  $\phi(L_1, 1) = L_2$ ,
- (2) For every  $t \in [0, 1]$ , the maps  $\phi_t$ 's are continues and injective, where  $\phi_t$  denotes to  $\phi$  restricted to  $\mathbb{R}^3 \times \{t\}$ .

The fact that  $L_1$  and  $L_2$  are ambient isotopic is denoted by  $L_1 \sim L_2$ .

From now on we denote, respectively, by  $\mathfrak{L}$  and  $|\mathfrak{L}|$  the set of all oriented links and unoriented links in  $\mathbb{R}^3$ . Moreover, for an oriented link  $L$ , we shall denote by  $|L|$  the unoriented link obtained by forgetting all orientation in  $L$ . Observe that if  $L_1$  and  $L_2$  are two oriented equivalent links, then  $|L_1|$  and  $|L_2|$  also are equivalent.

**Proposition 1.** *The relation  $\sim$  is an equivalence relation on both  $\mathfrak{L}$  and  $|\mathfrak{L}|$ .*

The main problem in knot theory is to find a subset of representatives of equivalence class of  $\mathfrak{L}$  and  $|\mathfrak{L}|$  under the relations  $\sim$ , or equivalently, given two links to decide if they are ambient isotopic or not. This problem is far from solution; however, we have a tool that helps to decide when two links are not ambient isotopic: the invariant of links.

**Definition 3.** *Let Set be a set, an invariant of oriented link is a function  $I : \mathfrak{L} \longrightarrow \text{Set}$  such that:*

$$\text{If } L_1 \sim L_2, \quad \text{then } I(L_1) = I(L_2).$$

*In the case that Set is a polynomial ring or a field of rational functions, we say that I is a polynomial invariant of links.*

To have an invariant when Set is well understood allows to compare easily the images of the links by I; therefore, I is useful in the sense that if the images of the links are different then the links are not equivalent. Some famous classical invariants are: the linking number, the 3-coloration, the fundamental group of a link and the following polynomial invariants:

- (1) The Alexander polynomial [9],
- (2) The Jones polynomial [24],
- (3) The Homflypt polynomial [20],
- (4) The Kauffman polynomial [29].

The Alexander and Jones invariants are polynomials in one variable. The Homflypt and Kauffman polynomial are in two variables. The Jones polynomial can be obtained as a specialization both of the Kauffman and Homflypt polynomials; the Alexander polynomial can be obtained also as a specialization of the Homflypt polynomial, but not as a specialization of the normalized Kauffman polynomial. We note that all these polynomial invariants are defined for oriented links.

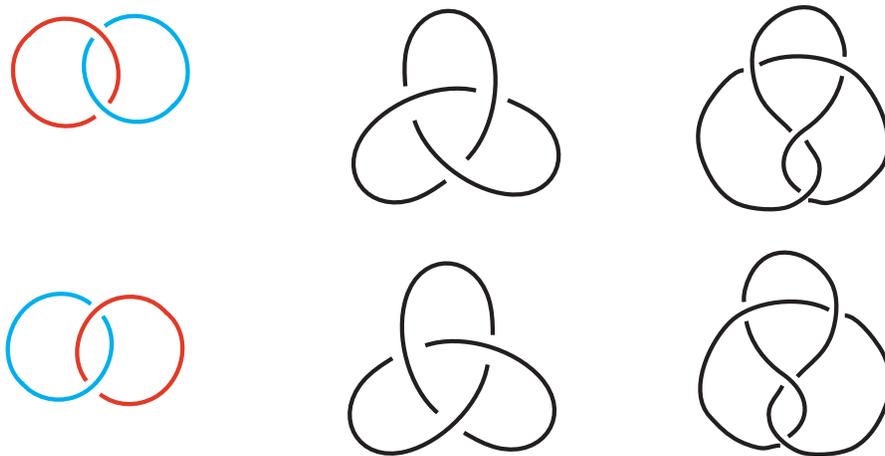
The problem of the classification of links contains the problems of the classification of subsets formed by certain kinds of links as: alternating, mutants and torus knots among others; all these problems are called global problems. The other types of problems concern the nature of a given link and are called local problems; one of the more basic and old local problem is to know if a link is equivalent to its mirror image.

mirror

**Definition 4.** Given a link  $L$ , oriented or not, its mirror image is the link  $L^*$  obtained by reflection of  $L$  with respect to a plane; technically,  $L^*$  can be considered as the image of  $L$  under the reflection  $(x, y, z) \mapsto (x, y, -z)$ .

**Definition 5.** If  $L$  and  $L^*$  are ambient isotopic we say that the links are amphicheiral, otherwise we say that the links are chiral.

**Example 1.** In the first row of the figure below we have, respectively, the Hopf link, the trefoil and the figure-eight. In the second row their respective reflected. The figure-eight are amphicheiral and the Hopf link and trefoil are chiral.



The amphicheirality of the Hopf link will be shown in Example 10 and for the amphicheirality of the Figure-eight see [1, p. 14]. The proof

that the Trefoil is cheiral was done in 1914 by Max Dhen, see [17]; his proof is a procedure known now as Dhen surgery. A remarkable fact, is that the cheirality of the trefoil is detected also by the Jones polynomial; in xx we show this result by using the Jones polynomial in its Kauffman version.

**Example 2.** Denote by  $H_+$  the oriented Hopf link with positive crossings. Below  $H_+$  and their mirror image.



In Example 9 we will prove that  $H_+$  is cheiral.

Others local problems are to determinate if a link is reversible and invertible.

**Definition 6.** The oriented link obtained by reversing every orientations in a link  $L$  is denoted by  $-L$ . A link is called reversible if  $L$  and  $-L$  are equivalent.

We can consider now  $-(L^*)$ , that is, the reversed of  $L^*$  and also  $(-L)^*$ , that is, the mirror image of  $-L$ ; however, observe that  $(-L)^* = -(L^*)$ .

**Definition 7.** An oriented link  $L$  is called invertible if  $L$  and  $(-L)^*$  are equivalent.

## 2. LINKS VIA DIAGRAMS

The links can be studied through diagrams in  $\mathbb{R}^2$ : we associate to each link a *generic projection* on a plane, this projection is provided with codes to indicate, at every double point, what portion of the projected curve (shadows) is coming from an over or under crossing. Thanks to a theorem of Kurt Reidemeister, the study of links can be

translated to the study of diagrams of links, allowing thus a combinatoric treating of the study of links.

**Definition 8.** *A generic projection of a curve of  $\mathbb{R}^3$  on a plane is one that only admits simple crossings.*

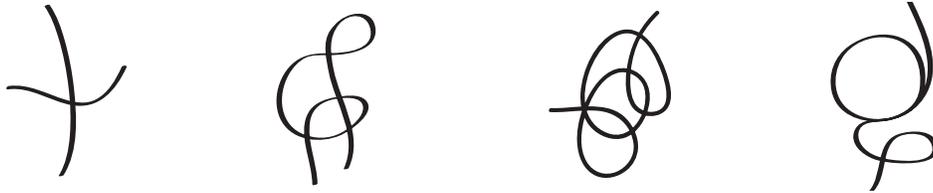


FIGURE 3

ProjectionMT

In Fig. 3, the first figure is a simple crossing, the second figure is a generic projection of a curve and the third and fourth figures are not generic projections of a curve.

**Definition 9.** *A diagram of a link is a generic projection (or shadow) of it on a plane, where each further simple crossing is codified, depending if it is originated from an over or under crossing in the link, by one of the two crossing codifications below.*

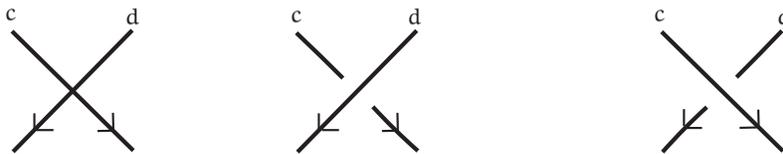


FIGURE 4

CodificationCrossing

More precisely, in Fig. 4, the first picture is a simple crossing in the diagram produced by the projection of the curves **c** and **d** of the link, the second picture shows the code used to indicate that the curve **c** is under the curve **d** in the link and the last picture is to indicate that the curve **c** is over the curve **d** in the link.

**Example 3.** Fig. 5 shows the procedure to obtain a projection of a left trefoil knot.

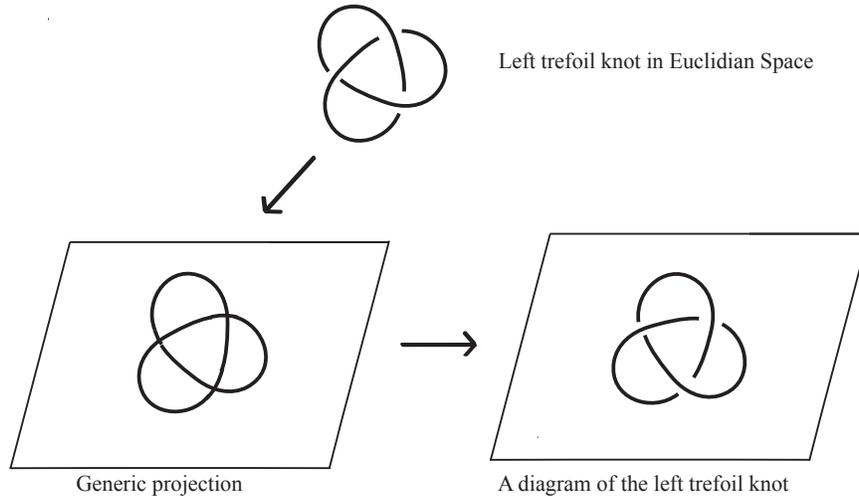


FIGURE 5

ProjectionTrefoil

Oriented links yields oriented diagrams; thus a diagram is oriented if every crossing codification is in one of the following situations:



FIGURE 6

DiagramCrossingOriented

The first codification is called (by convention) *positive crossing* and the other one *negative crossing*. The *sign of the crossing* is 1 if it is a positive crossing, otherwise is  $-1$ .

writhe

**Definition 10.** Let  $D$  be an oriented diagram, the writhe of  $D$ , denoted by  $w(D)$ , is the sum over the sign of all crossings of  $D$ :

$$w(D) := \sum \text{sig}(p),$$

where  $\mathfrak{p}$  running on the crossing of  $D$  and  $\text{sig}(\mathfrak{p})$  is the sign of the crossing  $\mathfrak{p}$ .

ReidemeisterMov

**Definition 11.** Two diagrams of unoriented links are  $R$ -isotopic if one of them can be transformed in the other, by so-called Reidemeister moves  $R_0, R_1, R_2$  and/or  $R_3$ , where in the case unoriented links are:

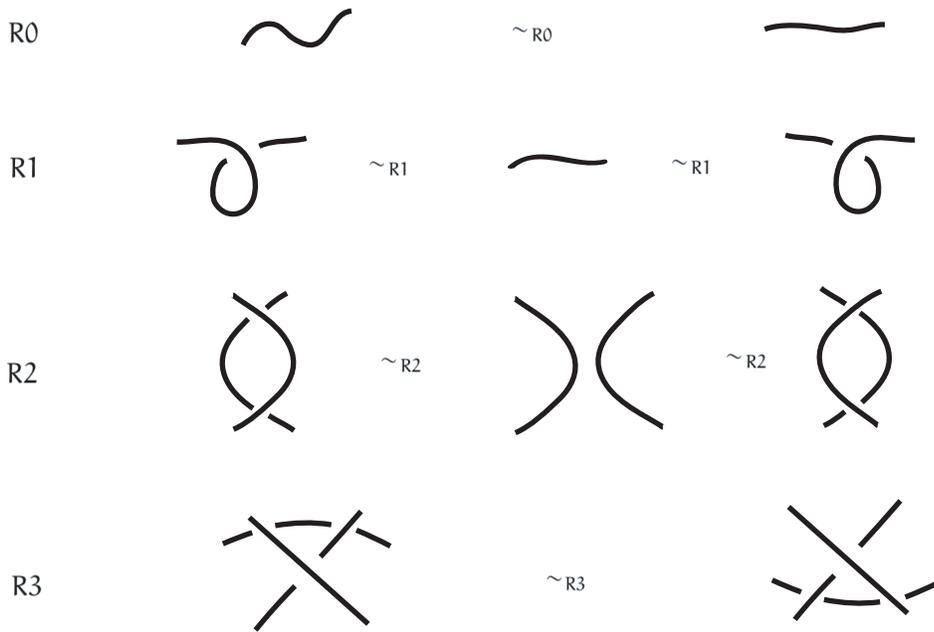


FIGURE 7

Reidemeister

We have the analogous definition for diagrams of oriented links but by adding now all possible orientations to the moves  $R_0$ – $R_3$  above. For instance, some of these oriented Reidemeister are shown in Fig. 8.

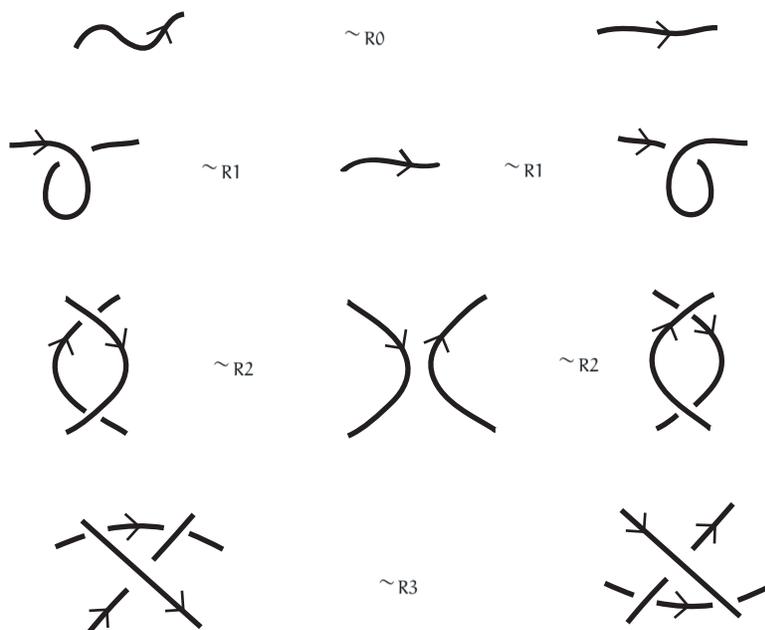


FIGURE 8

### OrientedReidemeister

**Proposition 2.** *The relation of  $R$ -isotopic, denoted by  $\sim_R$ , defines an equivalence relation on the set of planar diagrams of links.*

**Notation 1.** (1) *If  $D$  is an oriented diagram, we denote by  $|D|$  the unoriented diagram obtained by forgetting the orientation in  $D$ .*

(2) *We denote, respectively, by  $\mathfrak{D}$  and  $|\mathfrak{D}|$  the set of diagrams of, respectively, oriented and unoriented links in  $\mathbb{R}^3$ .*

### ReidemeisterTh

**Theorem 1** (Reidemeister, 1932). *Let  $L$  and  $L'$  be two links (oriented or not) and set  $D$  and  $D'$  diagrams, respectively, of  $L$  and  $L'$ , we have:*

$$L \sim L' \quad \text{if and only if} \quad D \sim_R D'.$$

*So,  $\mathfrak{L}/\sim$  is in bijection with  $\mathfrak{D}/\sim_R$ .*

### ExampleReidemeister

**Example 4.**

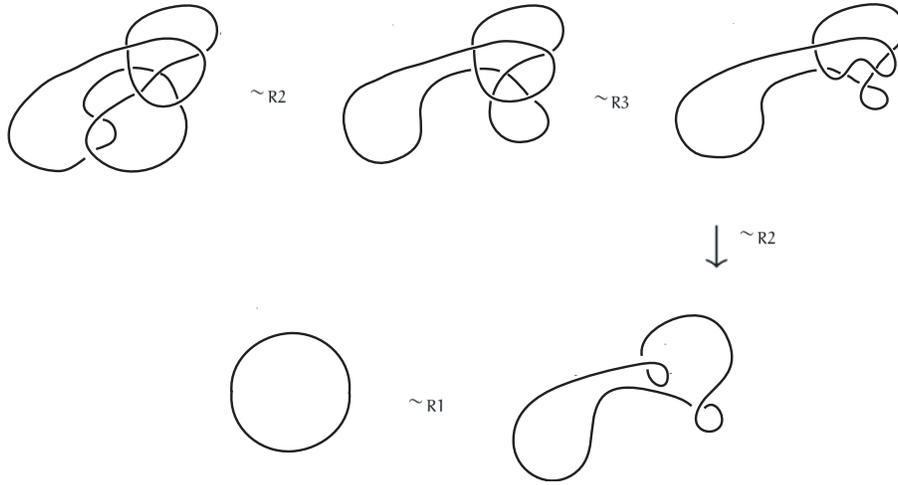
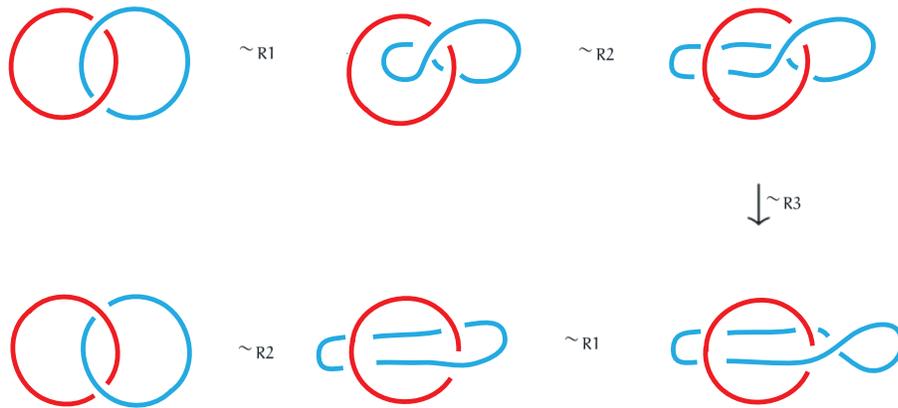


FIGURE 9

AmphicheiralHopfLink

**Example 5.** *The example below shows a way to prove that the Hopf link is amphicheiral:*



AmphicheiralHopfLink

FIGURE 10

The Reidemeister theorem says that constructing an invariant of links is equivalent to defining a function  $I : \mathcal{D} \rightarrow \text{Set}$ , such that it takes the same values on diagrams that differ in  $R_1, R_2$  and/or  $R_3$ .

**Definition 12.** *We say that  $I$  is an invariant of regular isotopy if the values of  $I$  does not change on unoriented links that are equal, up to the moves R2 and R3.*

wregularisotopy

**Proposition 3.**  *$w$  is an invariant of regular isotopy.*

*Proof.* To check that  $w$  agrees with the oriented move R2, it is enough to observe that introducing any orientation to , it turns out that the sum of the crossing signs is 0; the same happens with . Also, it is easy to see that by introducing any orientation to one of the crossing configurations of R3, the sum of the signs of the three crossings in it doesn't change after applying the move R3.  $\square$

The following lemma gives a recipe, due to L. Kauffman, to produce invariants of oriented links. This recipe was applied to define the Kauffman and Dubrovnik polynomials, see [29].

KauffmanRecipe

**Lemma 1** ([29, Lemma 2.1]). *Let  $\mathfrak{R}$  be a ring and  $\mathbf{a}$  an invertible element of  $\mathfrak{R}$ . Suppose that  $f$  is a regular isotopy invariant of unoriented diagram links taking values in  $\mathfrak{R}$ , such that:*

$$f(\overrightarrow{\circlearrowleft}) = \mathbf{a}f(\text{---}) \quad \text{and} \quad f(\overleftarrow{\circlearrowright}) = \mathbf{a}^{-1}f(\text{---}).$$

*Then, the function  $F$  defined as*

$$F(L) := \mathbf{a}^{-w(D)}f(|D|)$$

*is an invariant for oriented links, where  $D$  is a diagram for the oriented link  $L$ .*

### 3. BRACKET POLYNOMIAL

Let  $D$  be the diagram of a link. We will attach to  $D$  a polynomial in the variables  $A$ ,  $B$  and  $z$  by using the following inductive process:

- (1) We consider a crossing of  $D$  to assign the variables  $A$  and  $B$ :

AandBcrossing

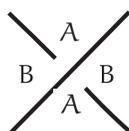


FIGURE 11

That is, we assign the variable  $A$  to the region obtained by sweeping the continuous arc in counterclockwise and the variable  $B$  to the remaining regions.

- (2) We smooth the crossing (1) by replacing it with the following two configurations:



FIGURE 12

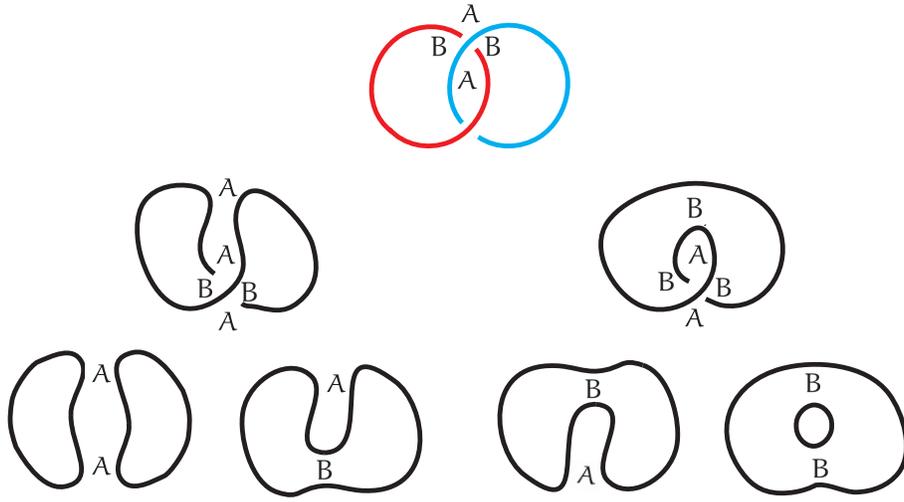
Fig9

We obtain thus two links, say  $D_1$  and  $D_2$ , to which we assign the marks  $A$  and  $B$  respectively.  $D_1$  and  $D_2$  have both a crossing number less than  $D$ .

- (3) We choose now a crossing in  $D_1$  and  $D_2$  and we apply again the process (2); then we obtain 4 diagrams with marks  $A$  and  $B$ . We apply again this process until we have only links without crossing.

Observe that if the original diagram has  $n$ -crossings, by applying (1)–(3) we obtain finally  $2^n$  diagrams, without crossings, having marks  $A$  and  $B$ . We call these  $2^n$  diagrams (with their marks) the *states* of  $D$ , the set of states of  $D$  is denoted by  $St(D)$ .

**Example 6.** *Below the procedure (1)–(3) described above for the Hopf link of Fig. 1.*



Given  $S \in \text{St}(\mathcal{D})$ , we denote by  $|S|$  the number of components of  $S$  and by  $\langle \mathcal{D}, S \rangle$  the product (commutative) of the marks appearing in  $S$ .

**Definition 13.** Let  $z$  be a variable commuting with  $A$  and  $B$ . The Bracket polynomial of an unoriented diagram  $\mathcal{D}$ , denoted by  $\langle \mathcal{D} \rangle$ , is the polynomial in  $\mathbb{Z}[A, B, z]$  defined by

$$\langle \mathcal{D} \rangle = \sum_{S \in \text{St}(\mathcal{D})} z^{|S|} \langle \mathcal{D}, S \rangle.$$

BracketHopf

**Example 7.** For the Hopf link, denoted by  $\mathcal{H}$ , we have

$$\text{St}(\mathcal{H}) = \left\{ \begin{array}{c} A \\ \text{---} \\ A \end{array} \text{---} \begin{array}{c} A \\ \text{---} \\ A \end{array}, \begin{array}{c} A \\ \text{---} \\ B \end{array} \text{---} \begin{array}{c} A \\ \text{---} \\ B \end{array}, \begin{array}{c} B \\ \text{---} \\ A \end{array} \text{---} \begin{array}{c} B \\ \text{---} \\ A \end{array}, \begin{array}{c} B \\ \text{---} \\ B \end{array} \text{---} \begin{array}{c} B \\ \text{---} \\ B \end{array} \right\}.$$

Thus, in this example it seems that  $|S|$  is the number of components  $-1$ , then

$$\langle \mathcal{D} \rangle = z^1 A^2 + z^0 AB + z^0 AB + z^1 B^2 = zA^2 + 2AB + zB^2.$$

**Proposition 4.** (1)  $\langle \bigcirc \sqcup \mathcal{D} \rangle = z \langle \mathcal{D} \rangle$ ,

$$(2) \langle \times \rangle = B \langle \text{---} \rangle + A \langle \rangle \langle \rangle,$$

$$(3) \langle \times \rangle = A \langle \text{---} \rangle + B \langle \rangle \langle \rangle.$$

*Proof.* See [30, Proposition 3.2]. □

We are going now to study the behaviour of the Bracket polynomial under the Reidemeister moves.

respectR1R2

- Lemma 2.** (1)  $\langle \text{R1} \rangle = AB \langle \text{R1} \rangle + (A^2 + B^2 + ABz) \langle \text{R1} \rangle$ ,  
 (2)  $\langle \text{R2} \rangle = (Az + B) \langle \text{R2} \rangle$ ,  
 (3)  $\langle \text{R3} \rangle = (Az + B) \langle \text{R3} \rangle$ .

*Proof.* See [30, Proposition 3.3]. □

According to (1) Lemma 2, in order to obtain that the Bracket polynomial respects the move R2 we must have:  $AB = 1$  and  $ABz + A^2 + B^2 = 0$ , or equivalently:

$$B = A^{-1} \quad \text{and} \quad z = -(A^2 + A^{-2}). \tag{1} \quad \text{conditionsABz}$$

**Proposition 5.** *Under the conditions of (1), the Bracket polynomial agrees with the Reidemeister move R3.*

*Proof.* See [30, Corollary 3.4]. □

**Proposition 6.** *Under the conditions of (1), we have:*

- (1)  $\langle \text{R2} \rangle = -A^3 \langle \text{R2} \rangle$ ,  
 (2)  $\langle \text{R3} \rangle = -A^{-3} \langle \text{R3} \rangle$ .

*Proof.* See [30, Corollary 3.4]. □

Denote by  $\langle \quad \rangle$  the function

$$\begin{aligned} \langle \quad \rangle : |\mathcal{D}| &\longrightarrow \mathbb{Z}[A, A^{-1}] \\ \mathbf{D} &\mapsto \langle \mathbf{D} \rangle \end{aligned}$$

This function is called function Bracket polynomial and is characterized in the following theorem.

**Theorem 2.** *The function Bracket polynomial  $\langle \quad \rangle$  is the unique function that satisfies:*

- (1)  $\langle \bigcirc \rangle = 1$ ,  
 (2)  $\langle \bigcirc \sqcup \mathbf{D} \rangle = -(A^{-2} + A^2) \langle \bigcirc \rangle$ ,  
 (3)  $\langle \text{R1} \rangle = A \langle \text{R1} \rangle + A^{-1} \langle \text{R1} \rangle$ .

**3.1.** Let  $D$  be a diagram of the oriented link  $L$ , we define  $f(L)$  as follows:

$$f(L) := (-A^3)^{-w(D)} \langle |D| \rangle \in \mathbb{Z}[A, A^{-1}].$$

Then, thanks to the Lemma 1, we have the following theorem.

JonesLAKauffman

**Theorem 3** ([29, Proposition 2.5]). *The map  $f : L \mapsto f(L)$  is an invariant of oriented links.*

f(H+)

**Example 8.** *For the oriented Hopf link,  $H_+$ , of Example 7 we have:*

$$\langle |H_+| \rangle = zA^2 + 2 + zA^{-2} = -(A^{-2} + A^2)^2 + 2 = -A^4 - A^{-4}.$$

Now,  $w(H_+) = 2$ , then

$$f(H_+) = A^{-6}(-(A^{-2} + A^2)^2 + 2) = -A^{-10} - A^{-2}.$$

**Remark 1.** By making  $A = t^{-1/4}$ , we have that  $f(L)$  becomes the value of the original version of the seminal Jones polynomial. It is a very interesting point, that such an important modern mathematical object can be constructed in such a simple way.

Notice now that the diagrams of a link  $L$ , oriented or not, are in bijection with the diagrams of the mirror image  $L^*$  of  $L$ . More precisely, if  $L$  is a unoriented link, the diagrams of  $L^*$  correspond those obtained by interchanging, in the diagrams of  $L$ , the crossing codification  $\times$  and  $\times$ . If  $L$  is an oriented the diagrams of  $L^*$  are obtained from the diagram of  $L$  by interchanging the crossing codification of Fig. 6.

**Proposition 7.** *Let  $L$  an oriented link and  $D$  a diagram of  $L$ , we have:*

- (1)  $\langle |D^*| \rangle$  is obtained by interchanging  $A$  with  $B$  in  $\langle |D| \rangle$ ,
- (2)  $f(L^*)$  is obtained by interchanging  $A$  with  $A^{-1}$  in  $f(L)$ .

The (2) of the proposition above says that  $f$  is useful to detect chirality; indeed,

**Corollary 1.** *If a link  $L$  is amphicheiral, then  $f(L)$  remains the same by changing  $A$  with  $A^{-1}$ .*

Hopf+Cheiral

**Example 9.** *The computation of Example 8 together with corollary above imply that  $H_+$  is cheiral.*

#### 4. LINKS VIA BRAIDS

Another way to study knot theory is through braids. Braids were introduced by E. Artin and the equivalence between braids and knots is due to two theorems: the Alexander and the Markov theorems. This section is a necessary compilation, for this exposition, on the equivalence between knot theory and braid theory.

Throughout these notes we denote by  $\mathfrak{S}_n$  the symmetric group on  $n$  symbols and we denote by  $s_i$  the elementary transposition  $(i, i + 1)$ . Recall that the Coxeter presentation of  $\mathfrak{S}_n$ , for  $n > 1$ , is that with generators  $s_1, \dots, s_{n-1}$  and the following relations:

$$\begin{aligned} s_i s_j &= s_j s_i \quad \text{for } |i - j| > 1, \\ s_i s_j s_i &= s_j s_i s_j \quad \text{for } |i - j| = 1. \end{aligned}$$

**4.1.** Let  $P_1, \dots, P_n$  be points in a fixed plane and  $Q_1, \dots, Q_n$  points in another plane parallel to the plane of the  $P_i$ 's. A  $n$ -geometrical braid is a collection of  $n$  arcs  $\alpha_1, \dots, \alpha_n$  connecting the initial points  $P_i$ 's with the ending points  $Q_{w(i)}$ , where  $w \in \mathfrak{S}_n$ , such that:

- (1) for every different  $i$  and  $j$  the arcs  $\alpha_i$  and  $\alpha_j$  are disjoint,
- (2) every plane parallel to the plane containing the points  $P_i$ 's meets each arc in only one point.

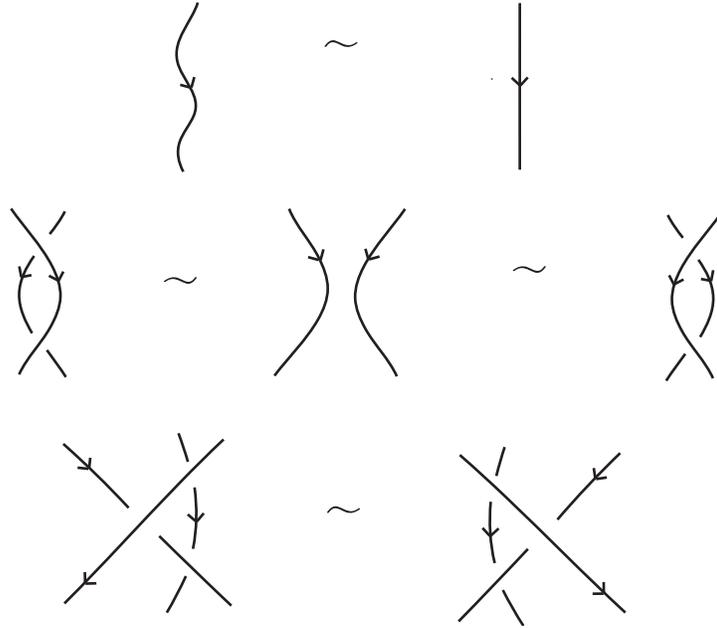
Geometrically, the arcs cannot be in the following situations:



Define  $\mathfrak{B}_n$  as the set formed by the  $n$ -geometrical braids. We define on  $\mathfrak{B}_n$  the equivalence relation, denoted by  $\approx$ , given by continuous deformation. That is, two geometrical braids  $\alpha$  and  $\beta$  are equivalent if there exists a family of geometrical braids  $\{\gamma_t\}_{t \in [0,1]}$  such that  $\gamma_0 = \alpha$  and  $\gamma_1 = \beta$ .

As in knot theory we can translate, equivalently, the geometrical braids to diagrams of them. More precisely, a *diagram of a geometrical braid* is the image of a generic projection of the braid in a plane. Thus, we have the analogous of Reidemeister theorem for braids.

**Theorem 4.** *Two geometrical braids are equivalent by  $\approx$  if and only if their diagrams are equivalent, that is, every diagram of one of them can be transformed in a diagram of the other, by using a finite number of times the following replacement:*



**Notation 2.** *We shall use also  $\mathfrak{B}_n$  to denote the set of diagrams of  $n$ -geometrical braids and  $\approx$  to denote the equivalence of diagrams of braids.*

Now, we can define the ‘product by concatenation’ between  $n$ -geometrical braids; more precisely, given  $\alpha$  and  $\beta$  in  $B_n$  we define by  $\alpha\beta \in \mathfrak{B}_n$  as that one defined by rescaling the result of the  $n$ -geometrical braid obtained by identifying the ending points of  $\alpha$  with the initial points of  $\beta$ .

rightdef

**Lemma 3.** For  $\alpha, \alpha', \beta$  and  $\beta'$  in  $\mathfrak{B}_n$  such that  $\alpha \approx \alpha'$  and  $\beta \approx \beta'$ , we have  $\alpha\beta \approx \alpha'\beta'$ .

*Proof.* We have  $\alpha\beta \approx \alpha'\beta$  and also  $\alpha'\beta \approx \alpha'\beta'$ , then  $\alpha\beta \approx \alpha'\beta'$ .  $\square$

Denote by  $B_n$  the set of equivalence class of  $\mathfrak{B}_n$  relative to  $\approx$ ; thus the elements of  $B_n$  are the equivalence classes  $[\alpha]$  of  $\alpha \in \mathfrak{B}_n$ . The Lemma 3 allows to pass the product by concatenation of  $\mathfrak{B}_n$  to  $B_n$ :

$$[\alpha][\beta] = [\alpha\beta] \quad (\alpha, \beta \in B_n).$$

**Theorem 5.**  $B_n$  is a group with the product by concatenation.

From now on we denote  $[\alpha]$  simply by  $\alpha$ . Thus, observe that the identity can be pictured by:

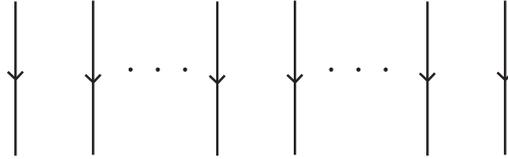


FIGURE 13

TrivialBraid

and the inverse  $\alpha^{-1}$  of  $\alpha$  is obtained as the mirror image with respect to the initial (or final) plane associated to  $\alpha$ . Fig. 14 shows the diagrams of  $\alpha = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^2 \in B_3$  and  $\alpha^{-1}$ .

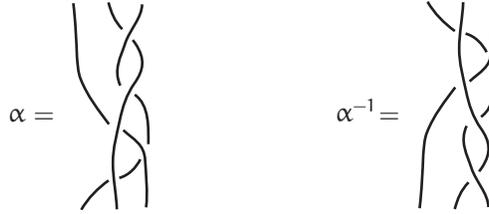


FIGURE 14

InverseBraid

**Remark 2.** Observe that  $B_1$  is the trivial group and  $B_2$  is the group of  $\mathbb{Z}$ .

For  $1 \leq i \leq n - 1$ , denote by  $\sigma_i$ 's the following *elementary braid*:

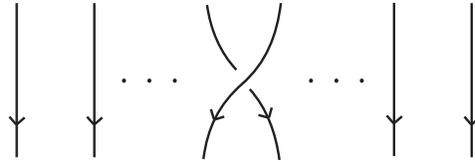


FIGURE 15

sigma\_i

**Theorem 6** (Artin, [10]). *For  $n > 1$ ,  $B_n$  can be presented by generators  $\sigma_1, \dots, \sigma_{n-1}$  and the following relations:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| > 1, \quad (2)$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i - j| = 1. \quad (3)$$

An immediate consequence of the above is that we have the following epimorphism:

$$\rho : B_n \longrightarrow \mathfrak{S}_n, \quad \text{defined by mapping } \sigma_i \mapsto s_i. \quad (4)$$

BnontoSn

Notice that for every  $n$  we have a natural monomorphism  $\iota_n$  from  $B_n$  in  $B_{n+1}$ , where for every braid  $\sigma \in B_n$ ,  $\iota_n(\sigma)$  is the braid of  $B_{n+1}$  coinciding with  $\sigma$  up to  $n^{\text{th}}$  strand and having one more strand with no crossing with the preceding strand.

**Notation 3.** We denote by  $B_\infty$  the group obtained as the inductive limit of  $\{(B_n, \iota_n)\}_{n \in \mathbb{N}}$ .

**4.2.** Given  $\alpha \in B_n$ , the identification of the initial points with the end points of  $\alpha$  determines a diagram of oriented links, which is denoted by  $\hat{\alpha}$ ; this process of identification is known as the closure of a braid. Thus we have the ‘function closure’

$$\hat{\cdot}: B_\infty \longrightarrow \mathfrak{L}, \quad \alpha \mapsto \hat{\alpha}.$$

The proof that the function closure is epijjective is due to Alexander [8]; we will outline this proof, since it gives an efficient method to compute the preimage, by  $\hat{\cdot}$ , of a given link.

Alexandertheorem

**Theorem 7** (J. W. Alexander, 1923). *Every oriented link is the closure of a braid.*

*Proof.* The sketch of the proof is as follows. Suppose we have a diagram  $D$  of an oriented link  $L$ .

- (1) We fix a point  $O$  in the plane not lying on any arc of  $D$ , such that a point moving along each component of the link is always seen from  $O$  going counterclockwise (or clockwise). Alexander proved that such a point  $O$  always exists in the isotopy class of the link diagram of  $D$ .
- (2) We divide the diagram in sectors, by rays starting from  $O$ , with the condition that each sector contains only one crossing.
- (3) Finally, we open the diagram along one of the rays obtaining a braid whose closure is the diagram  $D$ .

□

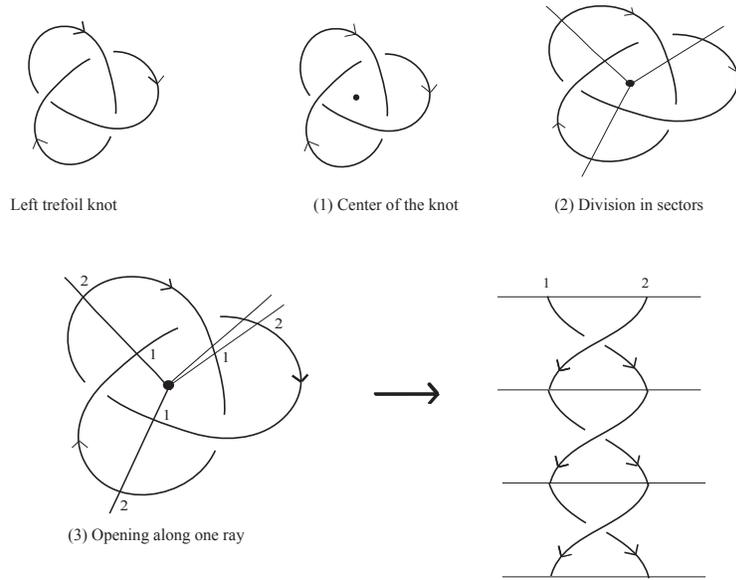


FIGURE 16

AlexanderTheorem

**Example 10.** *Fig. 16 shows that the Left-handed trefoil is the closure of the braid  $\sigma_1^3$ .*

In order to describe the links through braids we need to know when the closure of two braids yields the same link. In fact the map closure is not injective; for instance the braids  $1_{B_n}$  and  $\sigma_1 \cdots \sigma_n \in B_{n+1}$  yield the same link. The answer to which braids in  $B_\infty$  yield the same link is due to Markov [34], cf. [11].

Denote by  $\sim_M$  the equivalence relation on  $B_\infty$  generated by the following two replacements (also called Markov moves):

- (1) M1:  $\alpha\beta$  can be replaced by  $\beta\alpha$  (commutation),
- (2) M2:  $\alpha$  can be replaced by  $\alpha\sigma_n$  or by  $\alpha\sigma_n^{-1}$  (stabilization),

where  $\alpha$  and  $\beta$  are in  $B_n$ . Originally, Markov gave three moves and it was Weinberg [37] who showed that one of the Markov moves was superfluous. A

The relation  $\sim_M$  defines an equivalence relation in  $B_\infty$ . Two elements in the same  $\sim_M$ -class are called Markov equivalent.

MarkovTh

**Theorem 8** (A. A. Markov, 1936). *For  $\alpha$  and  $\beta$  in  $B_\infty$ , we have:  $\widehat{\alpha}$  and  $\widehat{\beta}$  are ambient isotopic links if and only if  $\alpha$  and  $\beta$  are Markov equivalent.*

From the Theorems 7 and 8, it follows that:

LinksAsBraids

**Corollary 2.** *There is a bijection between  $B_\infty / \sim_M$  and  $\mathcal{L} / \sim$  defined through the mapping  $\alpha \mapsto \widehat{\alpha}$ .*

Invariant

**Remark 3.** The Markov theorem above says that constructing an invariant for links is equivalent to defining a family of maps  $I = \{I_n\}_{n \geq 1}$ ,

$$I_n : B_\infty \longrightarrow \text{Set}$$

such that for all  $\alpha$  and  $\beta$  in  $B_n$  are fulfilled:

- (1)  $I_n(\alpha\beta) = I_n(\beta\alpha)$ ,
- (2)  $I_n(\alpha) = I_{n+1}(\alpha\sigma_n) = I_{n+1}(\alpha\sigma_n^{-1})$ .

**Remark 4.**  $L^*$  en terms of braids

## 5. HECKE ALGEBRAS

Let  $\mathbb{F}$  be a field, from now on the denomination  $\mathbb{F}$ -algebra means an associative unital, with unity 1, algebra over the field  $\mathbb{F}$ ; thus we can regard  $\mathbb{F}$  as a subalgebra of the center of the algebra.

Given a group  $G$ , we denote by  $\mathbb{F}G$  the  $\mathbb{F}$ -algebra known as the group algebra of  $G$  over  $\mathbb{F}$ . Recall that the set  $G$  is a linear basis for  $\mathbb{F}G$ , regarded as a  $\mathbb{F}$ -vector space. Also recall that if  $G$  has a presentation  $\langle X; R \rangle$ , then the  $\mathbb{F}$ -algebra  $\mathbb{F}G$  can be presented by generators  $X$  and the relations in  $R$ .

**5.1.** Let  $u$  be an indeterminate in  $\mathbb{C}$  and set  $\mathbb{K}$  the field of the rational functions  $\mathbb{C}(u)$ . For  $n \in \mathbb{N}$ , the Hecke algebra, denoted by  $H_n(u)$  or simply  $H_n$ , is defined by  $H_1 = \mathbb{K}$  and for  $n > 1$  as the  $\mathbb{K}$ -algebra

presented by generators  $h_1, \dots, h_{n-1}$  and the relations:

$$h_i h_j = h_j h_i \quad \text{for } |i - j| > 1, \quad (5)$$

$$h_i h_j h_i = h_j h_i h_j \quad \text{for } |i - j| = 1, \quad (6)$$

$$h_i^2 = u + (u - 1)h_i \quad \text{for all } i. \quad (7)$$

The  $h_i$ 's are invertible, indeed we have

$$h_i^{-1} = (u^{-1} - 1) + u^{-1}h_i. \quad (8)$$

inverseh

remarkHecke

**Remark 5.** (1) Taking  $u$  as power of a prime number, the Hecke algebra above appears in representation theory as a centralizer of a natural representation associated to the action of the finite general linear group on the variety of flags. This feature of the Hecke algebra will be the key point to construct here certain new invariants of links by using other Hecke algebra or other algebras of type Hecke. Consequently, the next subsection will be devoted to present the Hecke algebras in the context of representation theory capturing in particular the Hecke algebra defined above.

(2) The natural map  $\sigma_i \mapsto h_i$  defines an algebra epimorphism from  $\mathbb{K}B_n$  to  $H_n$ . Then, we have that the Hecke algebra  $H_n$  is the quotient of  $\mathbb{K}B_n$  by the two sided ideal generator by

$$\sigma_i^2 - u - (u - 1)\sigma_i \quad \text{for } 1 \leq i \leq n - 1.$$

(3) By taking the specialization  $u = 1$ , the Hecke algebra becomes the group algebra of the symmetric group. For this reason the Hecke algebra is known also as a deformation of the symmetric group.

We construct now a basis of the Hecke algebra; this basis is constructed in an inductive way and is used to prove that the algebra supports a Markov trace. We start with the following lemma.

atmost

**Lemma 4.** *In  $H_n$  every word in  $1, h_1, \dots, h_{n-1}$  can be written as a linear combination of words in the  $1$  and the  $h_i$ 's such that each of them contains at most one  $h_{n-1}$ . Hence,  $H_n$  is finite dimensional.*

*Proof.* The proof is by induction on  $n$ . For  $n = 2$  the lemma holds since  $H_2$  is the algebra generated by  $1$  and  $h_1$ . Suppose now that the lemma is valid for every  $H_k$ , with  $k < n + 1$ . We prove the lemma for  $n + 1$ ; let  $M$  be a word in  $1, h_1, \dots, h_n$  containing two times  $h_n$ , then we can write

$$M = M_1 h_n M_2 h_n M_3,$$

where  $M_i$ 's are words in  $1, h_1, \dots, h_{n-1}$ . But now, using the induction hypothesis we have to consider two situations according to  $M_2$  contains none or only one  $h_{n-1}$ . If  $M_2$  does not contain  $h_{n-1}$ , we have  $M_1 h_n M_2 h_n M_3 = M_1 h_n^2 M_2 M_3 = M_1 (u + (u - 1)h_n) M_2 M_3$ ; hence  $M = u M_1 M_2 M_3 + (u - 1) M_1 h_n M_2 M_3$ , thus  $M$  is reduced as the lemma claims. On the other hand, if  $M_2$  contains only one  $h_{n-1}$ , we can write it as  $M_2 = M' h_{n-1} M''$ , where  $M'$  and  $M''$  are words in  $1, h_1, \dots, h_{n-2}$ ; so  $M = M_1 h_n (M' h_{n-1} M'') h_n M_3$ ; by applying now (5) and (6) we obtain  $M = M_1 M' h_{n-1} h_n h_{n-1} M'' h_n M_3$ , then  $M$  is as the lemma claims.

In the case that  $M$  contains more than two generators  $h_n$ , we reduce two of them using the argument above; so arguing inductively we deduce that  $M$  can be written as the lemma is claiming.  $\square$

In  $H_n$ , define:  $C_1 = \{1, h_1\}$  and  $C_i = \{1, h_i x; x \in C_{i-1}\}$ , for  $2 \leq i \leq n - 1$ .

**Definition 14.** *The elements  $n_1 n_2 \cdots n_{n-1} \in H_n$ , with  $n_i \in C_i$ , are called normal words. This set formed by the normal words will be denoted by  $C_n$ .*

Observe that:

$$C_n = C_{n-1} \cup \{x h_{n-1} h_{n-2} \cdots h_i; x \in C_{n-1}, 1 \leq i \leq n - 1\}; \quad (9) \quad \square_{Cn}$$

hence  $\mathcal{C}_n$  consist of  $n!$  elements. Note that we have a natural bijection between  $\mathcal{C}_n$  and the symmetric group  $\mathbf{S}_n$ , since:

$$\mathbf{S}_n = \mathbf{S}_{n-1} \cup \{\mathbf{s}\mathbf{s}_{n-1}\mathbf{s}_{n-2} \cdots \mathbf{s}_i; \mathbf{s} \in \mathbf{S}_{n-1}, 1 \leq i \leq n-1\}. \quad (10)$$

normalSn

basisHecke

**Theorem 9.** *The set  $\mathcal{C}_n$  is a linear basis of  $H_n$ . Hence, the dimension of  $H_n$  is  $n!$ .*

*Proof.* We will prove, by induction on  $n$ , that  $\mathcal{C}_n$  is a spanning set of  $H_n$ . For  $n = 2$  the theorem is clear. Suppose now that the theorem is true for every  $k < n$ . From Lemma 4 it follows that  $H_n$  is linearly spanned by the elements of the form (i) and (ii):

$$(i) M_0 \quad \text{and} \quad (ii) M_1 h_{n-1} M_2,$$

where  $M_i$ 's are words in  $1, h_1, \dots, h_{n-2}$ . By the induction hypothesis, follows that  $M_i$ 's are linear combination of elements of  $\mathcal{C}_{n-1}$ , so we can suppose that  $M_i$ 's belong to  $\mathcal{C}_{n-1}$ . Thus, it is enough to prove that the elements of (ii) are a linear combination of the elements of  $\mathcal{C}_n$  (notice that  $M_0 \in \mathcal{C}_n$ ). Set  $M_2 = n_1 n_2 \cdots n_{n-2}$ , where  $n_i \in \mathcal{C}_i$ ; we have

$$M_1 h_{n-1} M_2 = M_1 h_{n-1} n_1 n_2 \cdots n_{n-2} = M_1 n_1 n_2 \cdots h_{n-1} n_{n-2}.$$

From the induction hypothesis  $M_1 n_1 n_2 \cdots$  is a linear combination of elements of  $\mathcal{C}_{n-1}$  and notice that  $h_{n-1} n_{n-2} \in \mathcal{C}_{n-1}$ . So, having in mind (9), we deduce that the elements in (ii) belong to the linear span of  $\mathcal{C}_n$ .

To end the prove, we will see that  $\mathcal{C}_n$  is a  $\mathbb{K}$ -linear independent set. For  $i \in \{1, \dots, n-1\}$ , we define the endomorphism  $L_i$  of  $\mathbb{K}\mathbf{S}_n$  by

$$L_i(\mathbf{s}) = \begin{cases} \mathbf{s}_i \mathbf{s} & \text{if } l(\mathbf{s}_i \mathbf{s}) > l(\mathbf{s}), \\ \mathbf{u} \mathbf{s}_i \mathbf{s} + (\mathbf{u} - 1) \mathbf{s} & \text{if } l(\mathbf{s}_i \mathbf{s}) < l(\mathbf{s}), \end{cases}$$

where  $l$  denotes the length on  $\mathbf{S}_n$  relative to the transposition generators  $\mathbf{s}_i$ 's, that is:  $l(\mathbf{s})$  is the minimal number of  $\mathbf{s}_i$ 's used to write  $\mathbf{s}$ . The mapping  $h_i \mapsto L_i$  defines an algebra homomorphism  $L$ , from  $H_n$  in  $\text{End}(\mathbb{K}\mathbf{S}_n)$ , for details see [16]. Notice that

$$L(h_j h_{j-1} \cdots h_i)(1) = \mathbf{s}_j \cdots \mathbf{s}_i \quad \text{for all } i < j. \quad (11)$$

Lh1

Now, the equation  $\sum_{x \in \mathcal{C}_n} \alpha_x x = 0$ , with  $\alpha_x \in \mathbb{K}$ , implies

$$\sum_{x \in \mathcal{C}_n} \alpha_x L(x)(1) = 0.$$

This equation can be rewritten as  $\sum_{x \in \mathcal{C}_n} \alpha_x \mathbf{s}_x = 0$ , where  $\mathbf{s}_x := L(\mathbf{n}_1)(1) \cdots L(\mathbf{n}_{n-1})(1)$  for  $x = \mathbf{n}_1 \cdots \mathbf{n}_{n-1}$ , where  $\mathbf{n}_i \in \mathcal{C}_i$ . But, (10) and (11) say that the  $\mathbf{s}_x$ 's are running the symmetric group  $\mathcal{S}_n$ , therefore  $\alpha_x = 0$ , for all  $x$ . □

The above theorem and (9) imply that we have a natural tower of algebras

$$H_1 = \mathbb{K} \subset H_2 \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots$$

We will denote by  $H_\infty$  the inductive limit associated to this tower.

Notice that the inclusion of algebras  $H_n \subset H_{n+1}$  allows to obtain a structure of  $(H_n, H_n)$ -bimodule for  $H_{n+1}$ ; further, we can consider the  $(H_n, H_n)$ -bimodule  $H_n \otimes_{H_{n-1}} H_n$  since  $H_n$  is a  $(H_n, H_{n-1})$ -bimodule and also a  $(H_{n-1}, H_n)$ -bimodule.

atmostn!n

**Proposition 8.** *The dimension of the  $\mathbb{K}$ -vector space  $H_n \otimes_{H_{n-1}} H_n$  is at most  $n!n$ .*

*Proof.* Theorem 9 implies that every element in  $H_n \otimes_{H_{n-1}} H_n$  is a  $\mathbb{K}$ -linear combination of elements of the form  $\mathbf{a} \otimes \mathbf{b}$ , where  $\mathbf{a}, \mathbf{b} \in \mathcal{C}_n$ . Now, we have two possibilities:  $\mathbf{b}$  is in  $\mathcal{C}_{n-1}$  or  $\mathbf{b} = x\mathbf{h}_{n-1} \cdots \mathbf{h}_i$ , with  $x \in \mathcal{C}_{n-1}$  (see (9)). Now, if  $\mathbf{b} \in \mathcal{C}_{n-1}$ , we have  $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b} \otimes \mathbf{1}$  and in the other case we can write  $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}x \otimes \mathbf{h}_{n-1} \cdots \mathbf{h}_i$ . Therefore, every element of  $H_n \otimes_{H_{n-1}} H_n$  is a linear combination of elements of the form  $\mathbf{a} \otimes \mathbf{1}$  and  $\mathbf{a} \otimes \mathbf{h}_{n-1} \cdots \mathbf{h}_i$ , where  $\mathbf{a} \in \mathcal{C}_n$  and  $1 \leq i \leq n - 1$ . Hence the proof follows. □

The following lemma will be used in the next section.

bimoduleHecke

**Lemma 5.** *The map  $\phi : H_n \oplus H_n \otimes_{H_{n-1}} H_n \longrightarrow H_{n+1}$ , defined by*

$$x + \sum_i y_i \otimes z_i \mapsto x + \sum_i y_i \mathbf{h}_n z_i$$

is an isomorphism of  $(H_n, H_n)$ -bimodules.

*Proof.* Every element  $c$  of  $H_{n-1}$  commutes with  $h_n$ , so  $ach_nb = ah_ncb$ , for all  $a, b \in H_n$ , that is  $\phi(ac \otimes b) = \phi(a \otimes ub)$ ; thus  $\phi$  is well defined. It is clear that  $\phi$  is a homomorphism of bimodules. Now, the Theorem 9 and Proposition 8 imply that the dimension of  $H_n \oplus H_n \otimes_{H_{n-1}} H_n$  is at most  $n! + n!n = (n+1)!$ . This fact, together with the fact that the dimension of  $H_{n+1}$  is also  $(n+1)!$  and that  $\phi$  is epijjective (see Lemma 4) allows to deduce that  $\phi$  is an isomorphism of bimodules.  $\square$

**5.2.** From now on  $z$  denotes a new variable commuting with  $u$ .

Ocneanu

**Theorem 10** (Ocneanu). *There exists a unique family of linear maps  $\tau = \{\tau_n\}_{n \in \mathbb{N}}$ , where  $\tau_n : H_n \rightarrow \mathbb{K}(z)$  is defined inductively by the following rules:*

- (1)  $\tau_1(1) = 1$ ,
- (2)  $\tau_n(ab) = \tau_n(ba)$ ,
- (3)  $\tau_{n+1}(ah_nb) = z\tau_n(ab)$ ,

where  $a, b \in H_n$ .

*Proof.* The definition of  $\tau_n$  is based on the homomorphism  $\phi$  of Lemma 5. For  $n = 1$ ,  $\tau_1$  is defined as the identity on  $\mathbb{K}$ . Now, given  $a \in H_{n+1}$ , the Lemma 5 says that, we can write uniquely,  $a = \phi(x + \sum_i y_i \otimes z_i)$ , then we define  $\tau_{n+1}$  by

$$\tau_{n+1}(a) := \tau_n(x) + z \sum_i \tau_n(y_i z_i).$$

For instance  $\tau_2(h_1) = z$ , since  $\phi(0 + 1 \otimes 1) = h_1$ .

For every  $a, b \in H_n$ , we have  $\phi(a \otimes b) = ah_nb$ , hence  $\tau_{n+1}(ah_nb) = z\tau_n(ab)$ , i.e. the rule (3) is satisfied.

We are going to check now the rule (2). The checking is by induction on  $n$ . For  $n = 2$  the rule is trivially satisfied. Suppose now that the rule (2) is true for  $n$ . Due to the linearity of  $\tau_n$  and Lemma 4, it is enough to prove that  $\tau_{n+1}(ab) = \tau_{n+1}(ba)$ , for  $a$  and  $b$  words having at most one  $h_n$ . In the case that  $h_n$  does not appear in  $a$  and  $b$  the prove follows

from the induction hypothesis. Suppose now the case that  $\mathbf{a}$  contains  $h_n$  and not  $\mathbf{b}$ ; put  $\mathbf{a} = xh_ny$ , where  $x$  and  $y$  are words in  $1, h_1, \dots, h_{n-1}$ . We have  $\tau_{n+1}(\mathbf{ab}) = \tau_{n+1}(xh_nyb) = z\tau_n(xyb)$ , by using the induction hypothesis on this last term, we obtain then  $\tau_{n+1}(\mathbf{ab}) = z\tau_n(\mathbf{bxy})$ , so  $\tau_{n+1}(\mathbf{ab}) = \tau_{n+1}(\mathbf{bxh_ny}) = \tau_{n+1}(\mathbf{ba})$ . Finally, we will see the case where both  $\mathbf{a}$  and  $\mathbf{b}$  contain one  $h_n$ : put  $\mathbf{a} = xh_ny$  and suppose  $\mathbf{b} = h_n$ , where  $x, y$  are words in  $1, h_1, \dots, h_{n-1}$  containing at most one  $h_{n-1}$ ; here we must analyse several cases depending whether  $h_{n-1}$  in  $x$  and  $y$  appears or not, but we will consider only the case more complicated case, that is, the case in which  $x$  and  $y$  contain  $h_{n-1}$ ; write  $x = x'h_{n-1}x''$  and  $y = y'h_{n-1}y''$ , where  $x', x'', y'$  and  $y''$  belong to  $H_{n-1}$ . We have:

$$\tau_{n+1}(\mathbf{ab}) = \tau_{n+1}(xh_nyh_n) = \tau_{n+1}(xh_n(y'h_{n-1}y'')h_n)$$

but  $h_n(y'h_{n-1}y'')h_n = y'h_nh_{n-1}h_ny'' = y'h_{n-1}h_nh_{n-1}y''$ , then

$$\begin{aligned} \tau_{n+1}(\mathbf{ab}) &= z\tau_n(xy'h_{n-1}^2y'') \\ &= z(u\tau_n(xx'y'') + (u-1)\tau_n(xy'h_{n-1}y'')). \end{aligned}$$

Hence

$$\tau_{n+1}(\mathbf{ab}) = z(u\tau_n(xy'y'') + (u-1)\tau_n(xy)). \quad (12) \quad \text{prooftrace1}$$

On the other side,

$$\tau_{n+1}(\mathbf{ab}) = \tau_{n+1}(h_nxh_ny_n) = \tau_{n+1}(h_nx'h_{n-1}x''h_ny_n);$$

now,  $h_nx'h_{n-1}x''h_ny_n = x'h_nh_{n-1}h_nx''y_n = x'h_{n-1}h_nh_{n-1}x''y_n$ , then

$$\tau_{n+1}(\mathbf{ab}) = z\tau_{n-1}(x'h_{n-1}^2x''y_n),$$

so, we obtain

$$\tau_{n+1}(\mathbf{ab}) = z(u\tau_n(x'x''y) + (u-1)\tau_n(xy)). \quad (13) \quad \text{prooftrace2}$$

From (12) and (13) it only remains to prove that  $\tau_n(x'x''y) = \tau_n(xy'y'')$ .

This is done as follows:

$$\begin{aligned} \tau_n(x'x''y) &= \tau_n(x'x''y'h_{n-1}y'') = z\tau_{n-1}(x'x''y'y'') \\ &= \tau_n(x'h_{n-1}x''y'y'') = \tau_n(xy'y''). \end{aligned}$$

□

tauhcubo

**Example 11.** To compute  $\tau(\mathfrak{h}_i^3)$ , we first note that  $\mathfrak{h}_i^3 = \mathfrak{h}_i \mathfrak{h}_i^2 = \mathfrak{h}_i(u + (u - 1)\mathfrak{h}_i)$ , so  $\mathfrak{h}_i^3 = u(u - 1) + (u^2 - u + 1)\mathfrak{h}_i$ . Then

$$\tau(\mathfrak{h}_i^3) = u(u - 1) + z(u^2 - u + 1).$$

## 6. THE HOMFLYPT POLYNOMIAL

sectionHomflypt

We show now the construction of the Homflypt polynomial due to V. Jones. This Jones construction gives a method (or Jones recipe) which is our main tool to construct invariants.

**6.1.** We have a natural representation from  $B_n$  in  $H_n$ , defined by mapping  $\sigma_i$  in  $\mathfrak{h}_i$ , however we need to consider a slightly more general representation, denoted by  $\pi_\theta$  and defined by mapping  $\sigma_i$  in  $\theta\mathfrak{h}_i$ , where  $\theta$  is a scalar factor; the reason for taking this factor  $\theta$  will be clear soon. Now, composing  $\pi_\theta$  with the Markov trace  $\tau_n$ , we have the maps

$$\tau_n \circ \pi_\theta : B_n \longrightarrow \mathbb{K}(z) \quad (n \in \mathbb{N}).$$

This family of maps yields a unique map  $X_\theta$  from  $B_\infty$  to  $\mathbb{K}(z)$ . Now, according to Remark 3, the function  $X_\theta$  defines an invariant of links, if it agrees with the replacements of Markov M1 and M2, or equivalently, for every  $n$  the function  $\tau_n \circ \pi_\theta$  agrees with M1 and M2. The fact that  $\pi_\theta$  is a homomorphism and the rule (2) of the Ocneanu trace implies that, for every  $n$  and  $\theta$  the maps  $\tau_n \circ \pi_\theta$  agree with the Markov replacement M1. For the replacement M2 we note that, in particular, the maps  $\tau_n \circ \pi_\theta$  must satisfy:

$$(\tau_n \circ \pi_\theta)(\sigma_{n-1}) = (\tau_n \circ \pi_\theta)(\sigma_{n-1}^{-1}) \quad \text{for all } n.$$

We have  $(\tau_n \circ \pi_\theta)(\sigma_{n-1}) = \theta z$  and from (8), we get:

$$(\tau_n \circ \pi_\theta)(\sigma_{n-1}^{-1}) = \theta^{-1} \tau_n(\mathfrak{h}_{n-1}^{-1}) = \theta^{-1}((u^{-1} - 1) + u^{-1}z)$$

Then we derive that  $\theta z = \theta^{-1}((u^{-1} - 1) + u^{-1}z)$ , from where the factor  $\theta$  satisfies:

$$\lambda := \theta^2 = \frac{(u^{-1} - 1) + u^{-1}z}{z} = \frac{1 - u + z}{uz}, \quad (14) \quad \text{RelationLambdaZeta1}$$

or equivalently

$$z = -\frac{1 - u}{1 - \lambda u}. \quad (15) \quad \text{RelationLambdaZeta1}$$

So, extending the ground field  $\mathbb{K}(z)$  to  $\mathbb{K}(z, \sqrt{\lambda}) = \mathbb{C}(z, \sqrt{\lambda})$ , the family of maps  $\{\tau_n \circ \pi_{\sqrt{\lambda}}\}_{n \in \mathbb{N}}$  agrees with the Markov replacements M1 and M2. However, it is desirable that the invariant takes the values 1 on the unknot, that is, we want  $(\tau_n \circ \pi_{\sqrt{\lambda}})(\sigma) = 1$ , for every  $n$  and every  $\sigma$  whose closure is the unknot; notice that the unknot is the closure of the braid  $\sigma_1 \sigma_2 \cdots \sigma_{n-1}$ , for all  $n$ . So, we have:

$$(\tau_n \circ \pi_{\sqrt{\lambda}})(\sigma_1 \sigma_2 \cdots \sigma_{n-1}) = (\theta z)^{n-1} = (\sqrt{\lambda} z)^{n-1}. \quad (16) \quad \text{homflypFactor}$$

Then, we need to normalize  $\tau_n \circ \pi_{\sqrt{\lambda}}$  by  $(\sqrt{\lambda} z)^{n-1}$ .

Homflypt

**Theorem 11.** *Let  $L$  be an oriented link obtained as the closure of the braid  $\sigma \in B_n$ . We define  $X : \mathcal{L} \rightarrow \mathbb{C}(z, \sqrt{\lambda})$ , by*

$$X(L) := \left( \frac{1}{\sqrt{\lambda} z} \right)^{n-1} (\tau_n \circ \pi_{\sqrt{\lambda}})(\sigma). \quad (17) \quad \text{FormulaX1}$$

*Then  $X$  is an invariant of ambient isotopy for oriented links.*

*Proof.* Thanks to Corollary 2, we need only to check that:

$$(i) X(\widehat{\alpha \beta}) = X(\widehat{\beta \alpha}) \quad \text{and} \quad (ii) X(\widehat{\alpha}) = X(\widehat{\alpha \sigma_n}) = X(\widehat{\alpha \sigma_n^{-1}}),$$

where  $\alpha, \beta \in B_n$ . Clearly (i) holds. We are going to check now only the second equality of (ii), the checking of first equality is left to the

reader. We have:

$$\begin{aligned}
(\tau_{n+1} \circ \pi_{\sqrt{\lambda}})(\alpha \sigma_n^{-1}) &= \tau_{n+1}(\pi_\lambda(\alpha) \pi_\lambda(\sigma_n^{-1})) = (\sqrt{\lambda})^{-1} \tau_{n+1}(\pi_\lambda(\alpha) \mathfrak{h}_n^{-1}) \\
&= (\sqrt{\lambda})^{-1} \tau_{n+1}(\pi_\lambda(\alpha) ((u^{-1} - 1) + u^{-1} \mathfrak{h}_n)) \\
&= (\sqrt{\lambda})^{-1} ((u^{-1} - 1) + z u^{-1}) \tau_n(\pi_\lambda(\alpha)) \\
&= (\sqrt{\lambda})^{-1} \sqrt{\lambda} z \tau_n(\pi_\lambda(\alpha)) \quad (\text{see (14)}) \\
&= \tau_{n+1}(\pi_\lambda(\alpha \sigma_n)).
\end{aligned}$$

Hence, we obtain  $X(\widehat{\alpha \sigma_n}) = X(\widehat{\alpha \sigma_n^{-1}})$ . □

**Example 12.** Denote by  $\mathbb{T}$  the oriented Left-handed trefoil of Fig. 16. As  $\mathbb{T}$  is the closure of  $\sigma_1^3 \in \mathbb{B}_2$ , we have

$$X(\mathbb{T}) = \left( \frac{1}{\sqrt{\lambda z}} \right) \tau_2(\pi_{\sqrt{\lambda}}(\sigma_1^3)).$$

Now,  $\tau_2(\pi_{\sqrt{\lambda}}(\sigma_1^3)) = \sqrt{\lambda}^3 \tau(\mathfrak{h}_1^3) = \sqrt{\lambda}^3 (u(u-1) + (u^2 + u + 1)z)$ , see Example 11. Then  $X(\mathbb{T}) = z^{-1} \lambda^{-1} (u(u-1) + (u^2 + u + 1)z)$ , hence

$$X(\mathbb{T}) = z^{-1} \lambda u (u-1) + \lambda (u^2 + u + 1)$$

so, by using the expression in (14) for  $z$ , we obtain

$$X(\mathbb{T}) = \lambda u^2 + \lambda - \lambda^2 u^2.$$

We will finish the subsection giving other expression of the Homflypt polynomial. First, notice that by using the expression (15) for  $z$ , we get

$$\frac{1}{z \sqrt{\lambda}} = - \frac{1 - \lambda u}{(1 - u) \sqrt{\lambda}}. \quad (18) \quad \square$$

Second, for  $\sigma = \sigma_{i_1}^{m_1} \cdots \sigma_{i_k}^{m_k}$ , where the  $\sigma_{i_j}$ 's are the braid generators of  $\mathbb{B}_n$ , we define the exponent  $e(\sigma)$  of  $\sigma$  as the sum  $m_1 + \cdots + m_k$ . Third, denote by  $\pi$  the natural representation of  $\mathbb{B}_n$  in  $\mathbb{H}_n$ , that is,  $\pi$  is defined by the mapping  $\sigma_i \mapsto \mathfrak{h}_i$ . Thus, we have  $\pi_{\sqrt{\lambda}}(\sigma) = \sqrt{\lambda}^{e(\sigma)} \pi(\sigma)$ ; therefore the Homflypt polynomial can be rewritten as follows.

$$X(L) = \left( - \frac{1 - \lambda u}{(1 - u) \sqrt{\lambda}} \right)^{n-1} \sqrt{\lambda}^{e(\sigma)} (\tau_n \circ \pi)(\sigma), \quad (19) \quad \text{FormulaX2}$$

where  $L = \widehat{\sigma}$  and  $\sigma \in \mathbb{B}_n$ .

**6.2.** The Homflypt polynomial has a definition by skein rules. This definition is useful to calculate it and also to study its relations with other invariants such as the Jones polynomial and the Alexander polynomial.

Denote by  $L_+, L_-$  and  $L_0$  three oriented links with, respectively, diagrams  $D_+, D_-$  and  $D_0$ , which are different only inside a disk, where they are respectively placed, as shows Fig 17.

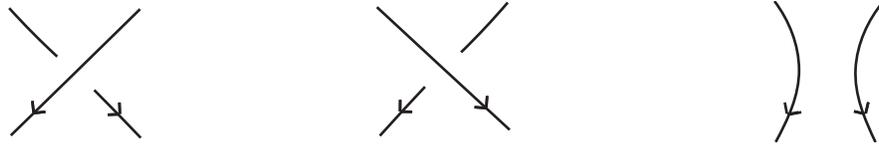


FIGURE 17

The links  $L_+, L_-$  and  $L_0$  are called a Conway triple; notice that in terms of braids, they can be written as:

$$L_+ = \widehat{\sigma\sigma_i}, \quad L_- = \widehat{\sigma\sigma_i^{-1}} \quad \text{and} \quad L_0 = \widehat{\sigma}$$

for some braid  $\sigma$  in  $B_n$ .

Now, keeping the notation above, we are going to compute the Homflypt polynomial on a Conway triple. First, set  $D = -\frac{1-\lambda u}{(1-u)\sqrt{\lambda}}$ , then we have:

$$\begin{aligned} X(L_-) &= D^{n-1}(\sqrt{\lambda})^{e(\sigma\sigma_i^{-1})}(\tau_n \circ \pi)(\sigma\sigma_i^{-1}) \\ &= D^{n-1}(\sqrt{\lambda})^{e(\sigma)}(\sqrt{\lambda})^{-1}\tau_n(\pi(\sigma)h_i^{-1}), \end{aligned}$$

by considering now (8), we obtain:

$$X(L_-) = D^{n-1}(\sqrt{\lambda})^{e(\sigma)}(\sqrt{\lambda})^{-1}((u^{-1} - 1)(\tau_n \circ \pi)(\sigma) + u^{-1}(\tau_n \circ \pi)(\sigma\sigma_i)).$$

Also a direct computation yields:

$$\begin{aligned} X(L_+) &= D^{n-1}(\sqrt{\lambda})^{e(\sigma)}\sqrt{\lambda}(\tau_n \circ \pi)(\sigma\sigma_i), \\ X(L_0) &= D^{n-1}(\sqrt{\lambda})^{e(\sigma)}(\tau_n \circ \pi)(\sigma). \end{aligned}$$

From these last three equations we obtain the following proposition.

SkeinX

**Proposition 9.** *The invariant  $X$  satisfies the following skein relation.*

$$\frac{1}{\sqrt{\lambda u}} X(L_+) - \sqrt{\lambda u} X(L_-) = \left( \sqrt{u} - \frac{1}{\sqrt{u}} \right) X(L_0).$$

SkeinHomflypt

**Theorem 12.** *There exists a unique function*

$$P : \mathfrak{L} \longrightarrow \mathbb{Z}[t, t^{-1}, x, x^{-1}]$$

such that:

- (1)  $P(\bigcirc) = 1$ ,
- (2)  $t^{-1}P(L_+) - tP(L_-) = xP(L_0)$  (*Skein relation*).

*Proof.* After a suitable change of variables,  $X$  satisfies the defining properties of  $P$ , so it remains to prove the uniqueness of  $P$ .  $\square$

**Remark 6.** Set  $l = it^{-1}$  and  $m = ix$ , where  $i$  is the complex number s.t.  $i^2 = -1$ . Then the skein relation of Theorem 12 can be rewritten as

$$lP(L_+) + l^{-1}P(L_-) = mP(L_0).$$

**Remark 7.** Making  $x = \sqrt{t} - 1/\sqrt{t}$  the polynomial  $P$  becomes the Jones polynomial, which is denoted by  $V$ . The Alexander polynomial, denoted by  $\Delta$ , is obtained by making  $t = 1$  and  $x = \sqrt{t} - 1/\sqrt{t}$ .

In the proposition below two of the main properties of the Homflypt polynomial are listed.

**Proposition 10.** *Let  $L$  be an oriented link, then:*

- (1)  $X(L) = X(-L)$ ,
- (2)  $X(L^*)$  is obtained by swapping, in  $X(L)$ ,  $\lambda$  by  $\lambda^{-1}$  and  $u$  by  $u^{-1}$ .

## 7. HECKE ALGEBRAS IN REPRESENTATION THEORY

**7.1.** Let  $G$  be a finite group. A complex representation of  $G$  is a pair  $(V, \rho)$ , where  $V$  is a finite dimensional space over  $\mathbb{C}$  and  $\rho$  is a homomorphism group from  $G$  to  $GL(V)$ . A subspace  $U$  of  $V$  is called  $G$ -stable if  $\rho_g(U) = U$ ; the representation is called irreducible if the unique  $G$ -stable subspaces are trivial. It is well known that every

complex representation can be decomposed as a direct sum of stable subspaces, see [35]. A fundamental problem in representation theory is: given a representation of  $G$ , write out such a decomposition. A powerful tool that helps the understanding of the decomposition of a representation is its centralizer, that is, the algebra formed by the endomorphisms of  $V$  commuting with  $\rho_g$ , for all  $g \in G$ . The centralizer of the representation  $(V, \rho)$  is denoted by  $\text{End}_G(V)$ .

Now, given a subgroup  $H$  of  $G$  we can construct the so called natural or induced representation, of  $G$  relative a  $H$ . More precisely, this representation can be made explicit as  $(\text{Ind}_H^G \mathbf{1}, \rho)$ , where:

$$\text{Ind}_H^G \mathbf{1} := \{f : G/H \longrightarrow \mathbb{C}; f \text{ is function}\}$$

and

$$\rho_g(f)(xH) = f(g^{-1}xH).$$

The centralizer of this representation is known as the Hecke algebra of  $G$  with respect to  $H$  and is usually denoted by  $\mathcal{H}(G, H)$ .

A fundamental piece in the theory of finite group of Lie type is the representation theory of the finite general linear group  $G = \text{GL}_n(\mathbb{F}_q)$ , where  $\mathbb{F}_q$  denotes the finite field with  $q$  elements; and an important family of irreducible representation of  $G$  appears in the decomposition of  $\text{Ind}_B^G \mathbf{1}$ , where  $B$  is the subgroup of  $G$  formed by the upper triangular matrices. The centralizer  $\mathcal{H}(G, B)$  of this representation was studied by N. Iwahori in the sixties, in a more general context for  $G$ : the finite Chevalley groups. In the case  $G$  is as above, that is, the finite general group, the algebra  $\mathcal{H}(G, B)$  corresponds to those of type  $A$  in the classification of Chevalley groups and the Iwahori theorem for  $\mathcal{H}(G, B)$  is the following.

Iwahori

**Theorem 13** (N. Iwahori, [21]). *The Hecke algebra  $\mathcal{H}(G, B)$  can be presented, as  $\mathbb{C}$ -algebra, by generators  $\phi_1, \phi_2, \dots, \phi_{n-1}$  and the following relations:*

$$(1) \phi_i^2 = q + (q - 1)\phi_i, \text{ for all } i,$$

- (2)  $\phi_i \phi_j = \phi_j \phi_i$ , for  $|i - j| > 1$ ,  
(3)  $\phi_i \phi_j \phi_i = \phi_j \phi_i \phi_j$ , for  $|i - j| = 1$ .

**7.2.** Now, we want to explain a little bit how the  $\phi_i$ 's look and work. To better explain we shall work in a more general setting. Let  $G$  be a finite group and set  $X$  a finite  $G$ -space. Define  $\mathbb{C}(X)$  the  $\mathbb{C}$ -vector space of all complex valued functions on  $X$ . Then, we have the representation  $(\mathbb{C}(X), \rho)$  of  $G$ , where  $(\rho_g f)(x) := f(g^{-1}x)$ . Consider now the  $G$ -space  $X \times X$  with action  $g(x, y) = (gx, gy)$  and the vector space  $\mathcal{K}(X) := \mathbb{C}(X \times X)$ , this vector space results to be an algebra with the convolution product:

$$I * J : (x, y) \mapsto \sum_{z \in X} I(x, z)J(z, y) \quad (I, J \in \mathcal{K}(X)).$$

**Theorem 14.** *The function  $\Phi$  from  $\mathcal{K}(X)$  to  $\text{End}(\mathbb{C}(X))$  is an algebra isomorphism, where  $\Phi : I \mapsto \Phi_I$ , is defined by*

$$(\Phi_I f)(x) := \sum_{y \in X} I(x, y)f(y) \quad (f \in \mathbb{C}(X), x \in X).$$

*Proof.* It is a routine to verify that  $\Phi$  is a linear map. Now, we have that  $\mathcal{K}(X)$  and  $\text{End}(\mathbb{C}(X))$  have the same dimension; thus, the bijectivity follows by proving that  $\Phi$  is injective. Let  $I \in \mathcal{K}(X)$  such that  $\Phi_I = 0$ ; then, in particular, we have  $\Phi_I(\delta_x) = 0$ , for all  $x \in X$ , where  $\delta_x$  denotes the Dirac function on  $X$ . This last equality implies  $I(x, y) = 0$ , for all  $(x, y) \in X \times X$ ; then  $I = 0$ , hence  $\Phi$  is injective.

To conclude the proof, we are going to prove that  $\Phi$  is an algebra homomorphism. Let  $I, J \in \mathcal{K}(X)$ , for all  $f \in \mathbb{C}(X)$  and  $x \in X$ , we have:

$$[(\Phi_I \circ \Phi_J)(f)](x) = \Phi_I(\Phi_J(f))(x) = \sum_{z \in X} I(x, z)\Phi_J(f)(z).$$

Then

$$[(\Phi_I \circ \Phi_J)(f)](x) = \sum_{z \in X} I(x, z) \sum_{y \in X} J(z, y)f(y) = \sum_{y \in X} \sum_{z \in X} I(x, z)J(z, y)f(y),$$

so

$$[(\Phi_I \circ \Phi_J)(f)](x) = \sum_{y \in X} (I * J)(x, y)f(y) = (\Phi_{I*J}f)(x).$$

Hence  $\Phi_I \circ \Phi_J = \Phi_{I*J}$ . □

Recall that  $\text{End}_G(\mathbb{C}(X))$  denote the centralizer of the representation  $(\mathbb{C}(X), \rho)$  and denote now by  $\mathcal{K}_G(X)$  the subalgebra of all  $I \in \mathcal{K}(X)$ , that are  $G$ -invariant, that is,  $I(gx, gy) = I(x, y)$ , for  $(x, y) \in X \times X$  and  $g \in G$ .

Now, set  $I \in \mathcal{K}(X)$  such that  $\Phi_I \in \text{End}_G(\mathbb{C}(X))$ , then

$$\rho_g \circ \Phi_I = \Phi_I \circ \rho_g \quad (g \in G).$$

So, in particular  $(\rho_g \circ \Phi_I)(\delta_x) = (\Phi_I \circ \rho_g)(\delta_x)$ , for all  $g \in G$  and  $x \in X$ ; hence

$$(\rho_g \circ \Phi_I)(\delta_x)(y) = (\Phi_I \circ \rho_g)(\delta_x)(y) \quad (x, y \in X, g \in G). \quad (20) \quad \text{RhoPhi=PhiRho}$$

We have

$$(\rho_g \circ \Phi_I)(\delta_x)(y) = (\rho_g(\Phi_I \delta_x))(y) = (\Phi_I \delta_x)(g^{-1}y) = \sum_{z \in X} I(g^{-1}y, z)\delta_x(z),$$

hence

$$(\rho_g \circ \Phi_I)(\delta_x)(y) = I(g^{-1}y, x). \quad (21) \quad \text{RhoPhi}$$

On the other hand,

$$\begin{aligned} (\Phi_I \circ \rho_g)(\delta_x)(y) &= \Phi_I(\rho_g \delta_x)(y) = \sum_{z \in X} I(y, z)(\rho_g \delta_x)(z) \\ &= \sum_{z \in X} I(y, z)\delta_x(g^{-1}z); \end{aligned}$$

so

$$(\Phi_I \circ \rho_g)(\delta_x)(y) = I(y, gx). \quad (22) \quad \text{PhiRho}$$

Thus, from (20)–(22) follows that  $I \in \mathcal{K}(X)$  is such that  $\Phi_I \in \text{End}_G(\mathbb{C}(X))$  is equivalent to state that  $I$  satisfies

$$I(x, y) = I(gx, gy) \quad (x, y \in X, g \in G).$$

Thus, we obtain the following theorem.

**Theorem 15.** *The algebra  $\mathcal{K}_G(X)$  is mapped, by the isomorphism  $\Phi$ , to  $\text{End}_G(\mathbb{C}(X))$ .*

Let  $O_1, \dots, O_m$  be the orbits of the  $G$ -space  $X \times X$ , hence the canonical basis of  $\mathcal{K}_G(X)$  is  $\{K_1, \dots, K_m\}$ , where the  $K_i$ 's are the function delta of Dirac,

$$K_i(x, y) := \begin{cases} 1 & \text{if } (x, y) \in O_i \\ 0 & \text{if } (x, y) \notin O_i. \end{cases}$$

**Corollary 3.** *A basis of  $\text{End}_G(\mathbb{C}(X))$  is the set  $\{\phi_1, \dots, \phi_m\}$ , where*

$$(\phi_i f)(x) = \sum_{y \text{ s.t. } (x, y) \in O_i} f(y).$$

*Hence, the dimension of  $\text{End}_G(\mathbb{C}(X))$  is the number of  $G$ -orbits of  $X \times X$ .*

*Proof.* Notice that  $\phi_i := \Phi(K_i)$ , so the proof is clear.  $\square$

**7.3.** We shall finish the section by showing the basis  $\phi_i$  in the case  $G = \text{GL}_n(\mathbb{F}_q)$  and  $X$  the variety of flags in  $\mathbb{F}_q^n$ .

remarkJones

**Remark 8.** In [25, p. 336] Jones raised the question whether his method of construction of the Homflypt polynomial can be used for Hecke algebras of other type. In this light it is natural to ask also whether the Jones method can be still used keeping  $G = \text{GL}_n(\mathbb{F}_q)$  but changing  $B$  for other notable subgroups of  $G$ . This is what we did to define new invariants for links. More precisely, we change  $B$  by his unipotent part, that is, the subgroup formed by the upper unitriangular matrices.

## 8. TEMPERLEY–LIEB ALGEBRAS

The Temperley–Lieb algebras were applied by Jones [24] to the construction of a polynomial invariant that revolutioned knot theory: the Jones polynomial. These algebras appear originally in the context of Stistical Mechanics [36] and were rediscovered by Jones when studying subfactors [23]. It was observed by Steinberg that the Temperley–Lieb

algebras can be realized as a quotient of the Hecke algebras; this is the point of view that we shall use here to describe them and their relation with the Jones polynomial briefly.

**8.1.** Recall that  $h_i$ 's denote the defining generators of the Hecke algebra  $H_n$ . Let  $s_{i_1} s_{i_2} \cdots s_{i_m}$  be a writing reduced of  $w \in \mathfrak{S}_n$ , then thanks to the Matsumoto Theorem [15, Theorem 64.20], we can define

$$h_w := h_{i_1} h_{i_2} \cdots h_{i_m} \in H_n.$$

For every elementary transpositions  $s_i$  and  $s_j$  with  $|i-j| = 1$ , we denote by  $H_{i,j}$  the subgroup of  $\mathfrak{S}_n$  generated by  $s_i$  and  $s_j$ , which is isomorphic to  $S_3$ . Further, for every  $i, j$  such that  $|i-j| = 1$ , we define

$$h_{i,j} = \sum_{w \in H_{i,j}} h_w. \tag{23} \quad \boxed{h_{i,j}}$$

**Definition 15.** *The Temperley–Lieb algebra, denoted by  $TL_n(u)$  or simply  $TL_n$ , is defined as the quotient of the Hecke algebra  $H_n$  by factoring out the relations  $h_{ij} = 0$ .*

**Remark 9.** *The elements  $h_{i,j}$ 's are conjugate to  $h_{1,2}$ . Then  $h_{1,2} = 0$ , implies  $h_{i,j} = 0$  for all  $i, j$ .*

**8.2.** Define  $f_i$  by

$$f_i = \frac{1}{1+u}(1+h_i) \quad (1 \leq i \leq n-1).$$

The  $f_i$ 's form a set of generators for  $H_n$  since  $f_i$  is a linear combination of defining generators of  $H_n$ . Moreover, we have the following proposition.

PresentationHn

**Proposition 11.** *The Hecke algebra  $H_n$  can be presented by the generators  $f_1, \dots, f_{n-1}$  with the following relations:*

$$f_i^2 = f_i \quad \text{for all } i, \tag{24}$$

$$f_i f_j = f_j f_i \quad \text{for } |i-j| > 1, \tag{25}$$

$$f_i f_j f_i - \frac{u}{(1+u)^2} f_i = f_j f_i f_j - \frac{u}{(1+u)^2} f_j \quad \text{for } |i-j| = 1. \tag{26}$$

*Proof.* The proof follows by checking that, respectively, the relations (24), (25) and (26) are equivalents to relations (7), (5) and (6). The checking is a routine, however we shall check that relations (26) and (6) are equivalents. Without loss of generality we can assume that  $i = 1$  and  $j = 2$ . We have  $h_i = (1 + u)^{-1}f_i - 1$ , this expression together with (24) shows that

$$h_1 h_2 h_1 = f_1 f_2 f_1 - \frac{u}{(1+u)^2} f_1, \quad h_2 h_1 h_2 = f_2 f_1 f_2 - \frac{u}{(1+u)^2} f_2.$$

So, the relations (6) and (26) are equivalents.  $\square$

Observe that from a direct computation, we obtain

$$h_{i,j} = f_i f_j f_i - \frac{u}{(1+u)^2} f_i, \quad \text{for } |i - j| = 1.$$

This fact and Proposition 11 implies the following proposition.

**Proposition 12.** *The Temperley–Lieb algebra  $TL_n(u)$  can be presented by generators  $f_1, \dots, f_{n-1}$  and the relations:*

$$f_i^2 = f_i \quad \text{for all } i, \quad (27)$$

$$f_i f_j = f_j f_i \quad \text{for } |i - j| > 1, \quad (28)$$

$$f_i f_j f_i = \frac{u}{(1+u)^2} f_i \quad \text{for } |i - j| = 1. \quad (29)$$

**8.3.** TL as a diagram algebra (Jones monoids) TL a deformation as Jones monoids.

## 9. YOKONUMA–HECKE ALGEBRAS

In this section we introduce the Yokonuma–Hecke whose origin is in representation theory of finite Chevalley groups; indeed, it appears as a centralizer of a natural representation of a Chevalley group. With the aim to apply this algebra to knot theory, we noted that it is linked, not only to the braid group, but also to the framed braid group. We show an inductive basis, the existence of a Markov trace and the E–system, all that fundamental ingredients to construct our invariants.

**9.1.** The Yokonuma–Hecke algebra comes from a centralizer of the permutation representation associated to the finite general linear group  $GL_n(\mathbb{F}_q)$  with respect to the subgroup consisting of the upper unitriangular matrix, cf. Remark 8. This centralizer was studied by T. Yokonuma who obtained a presentation of it, analogous to that found by N. Iwahori for the Hecke algebra, see Theorem 13. However, the presentation of Yokonuma was slightly modified in order to be applied to knot theory; a little modification of this presentation defines the so called Yokonuma–Hecke algebra.

**Definition 16.** *The Yokonuma–Hecke algebra, in short called Y–H algebra and denoted by  $Y_{d,n}(\mathbf{u})$ , is defined as follows:  $Y_{1,1}(\mathbf{u}) = \mathbb{K}$ , and for  $d, n \geq 2$  as the algebra presented by braid generators  $g_1, \dots, g_{n-1}$  and framing generators  $t_1, \dots, t_n$ , subject to the following relations:*

$$g_i g_j = g_j g_i \quad \text{for } |i - j| > 1, \quad (30)$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{for } |i - j| = 1, \quad (31)$$

$$t_i t_j = t_j t_i \quad \text{for all } i, j, \quad (32)$$

$$t_j g_i = g_i t_{s_i(j)} \quad \text{for all } i, j, \quad (33)$$

$$t_i^d = 1 \quad \text{for all } i, \quad (34)$$

$$g_i^2 = 1 + (u - 1)e_i(1 + g_i) \quad \text{for all } i, \quad (35)$$

where  $s_i = (i, i + 1)$  and

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}.$$

Whenever the variable  $\mathbf{u}$  is irrelevant, we shall denote  $Y_{d,n}(\mathbf{u})$  simply by  $Y_{d,n}$ .

remarkYn

**Remark 10.** (1) *The elements  $e_i$ 's result to be idempotents, i.e.  $e_i^2 = e_i$  and the  $g_i$ 's are invertible,*

$$g_i^{-1} = g_i + (u^{-1} - 1)e_i + (u^{-1} - 1)e_i g_i. \quad (36)$$

inversgi

*These facts will be frequently used in what follows.*

- (2) Notice that for  $\mathbf{d} = 1$ , the algebra  $Y_{\mathbf{d},\mathbf{n}}$  becomes  $H_{\mathbf{n}}$ , while for  $\mathbf{n} = 1$ ,  $Y_{\mathbf{d},\mathbf{n}}$  becomes the group algebra of the cyclic group with  $\mathbf{d}$  elements.
- (3) The mapping  $\mathbf{g}_i \mapsto \mathbf{h}_i$  and  $\mathbf{t}_i \mapsto 1$ , define an homomorphism algebra from  $Y_{\mathbf{d},\mathbf{n}}$  onto  $H_{\mathbf{n}}$ .

**9.2.** We have a natural representation of the braid group in the Y–H algebra since relations (i) and (ii) correspond to the defining relations of the braid group. Moreover, the relations (i)–(iv) correspond to the defining relations of the framing group. To be more precise, the framed braid group, denoted by  $\mathcal{F}_{\mathbf{n}}$ , is the semi–direct product  $\mathbb{Z}^{\mathbf{n}} \rtimes \mathbf{B}_{\mathbf{n}}$ , where the action that defines the semidirect product is through the homomorphism  $\mathbf{p}$  from  $\mathbf{B}_{\mathbf{n}}$  onto  $\mathbf{S}_{\mathbf{n}}$  (see (4)), that is:

$$\sigma(z_1, \dots, z_{\mathbf{n}}) = (z_{\sigma(1)}, \dots, z_{\sigma(\mathbf{n})}), \quad \text{where } \sigma(\mathbf{i}) := \mathbf{p}(\sigma)(\mathbf{i}).$$

In multiplicative notation the group  $\mathbb{Z}$  has the presentation  $\langle \mathbf{t}; - \rangle$  and the direct product  $\mathbb{Z}^{\mathbf{n}}$  has the presentation  $\langle \mathbf{t}_1, \dots, \mathbf{t}_{\mathbf{n}} ; \mathbf{t}_i \mathbf{t}_j = \mathbf{t}_j \mathbf{t}_i \text{ for all } \mathbf{i}, \mathbf{j} \rangle$ . Consequently,  $\mathcal{F}_{\mathbf{n}}$  can be presented by (braids) generators  $\sigma_1, \dots, \sigma_{\mathbf{n}-1}$  together with the (framing) generators  $\mathbf{t}_1, \dots, \mathbf{t}_{\mathbf{n}}$  subject to the relations (2), (3),  $\mathbf{t}_i \mathbf{t}_j = \mathbf{t}_j \mathbf{t}_i$ , for all  $\mathbf{i}, \mathbf{j}$  and the relations:

$$\mathbf{t}_j \sigma_{\mathbf{i}} = \sigma_{\mathbf{i}} \mathbf{t}_{\mathbf{s}_{\mathbf{i}}(\mathbf{j})} \quad \text{for all } \mathbf{i}, \mathbf{j}. \quad (37)$$

Now, because  $\mathcal{F}_{\mathbf{n}}$  is a semidirect product, we have that every element in it can be written in the form  $\mathbf{t}_1^{\mathbf{a}_1} \dots \mathbf{t}_{\mathbf{n}}^{\mathbf{a}_{\mathbf{n}}} \sigma$ , where  $\sigma \in \mathbf{B}_{\mathbf{n}}$  and the  $\mathbf{a}_i$ 's are integers called the framing. Further, observe that:

$$(\mathbf{t}_1^{\mathbf{a}_1} \dots \mathbf{t}_{\mathbf{n}}^{\mathbf{a}_{\mathbf{n}}} \sigma)(\mathbf{t}_1^{\mathbf{b}_1} \dots \mathbf{t}_{\mathbf{n}}^{\mathbf{b}_{\mathbf{n}}} \tau) = \mathbf{t}_1^{\mathbf{a}_1 + \mathbf{b}_{\sigma(1)}} \dots \mathbf{t}_{\mathbf{n}}^{\mathbf{a}_{\mathbf{n}} + \mathbf{b}_{\sigma(\mathbf{n})}} \sigma \tau. \quad (38)$$

productFn

In diagrams, the element  $\mathbf{t}_1^{\mathbf{a}_1} \dots \mathbf{t}_{\mathbf{n}}^{\mathbf{a}_{\mathbf{n}}} \sigma$  can be represented by the usual diagram braid for  $\sigma$  together with  $\mathbf{a}_1, \dots, \mathbf{a}_{\mathbf{n}}$  written at the top of the braid:  $\mathbf{a}_i$  is placed where the strand  $\mathbf{i}$  starts. For instance, Fig. 18 represents the framed braid  $\mathbf{t}_1^{\mathbf{a}} \mathbf{t}_2^{\mathbf{b}} \mathbf{t}_3^{\mathbf{c}} \mathbf{t}_4^{\mathbf{d}} \sigma_2 \sigma_3^{-1} \sigma_1^{-1} \sigma_2^2 \sigma_3$ .

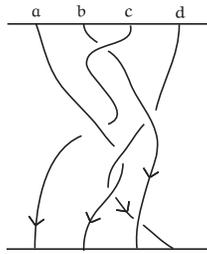


FIGURE 18

FramedBraid

In terms of diagrams, the formula (38) is translated as follows: we place the diagram of the braid  $\tau$  under the diagram of the braid  $\sigma$  and the framing  $\mathbf{b}_i$  travels along the strand up to the top of the diagram of  $\sigma\tau$ , so that the framings of the product are  $(\mathbf{a}_i + \mathbf{b}_{\sigma(i)})$ 's. For instance, for  $\sigma = \sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$  and  $\tau = \sigma_1\sigma_3^{-1}$ , the product  $(t_1^a t_2^b t_3^c t_4^d \sigma)(t_1^x t_2^y t_3^z t_4^w \tau)$  in terms of diagrams is showed in Fig. 19.

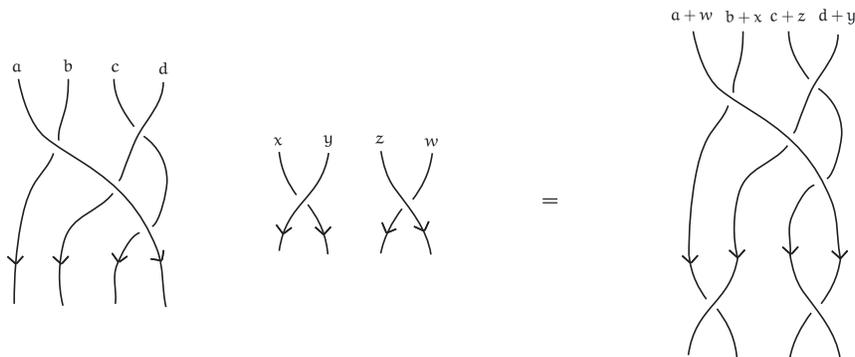


FIGURE 19

ProductFramedBraid

Finally, relations (i)–(v) correspond to the defining relations of the framing module  $\mathbf{d}$ . The  $\mathbf{d}$ -modular framed braid group  $\mathcal{F}_{\mathbf{d},n}$  is the semidirect product  $\mathbb{Z}/\mathbf{d}\mathbb{Z} \rtimes \mathbf{B}_n$ ; in other terms, it is the group obtained by imposing the relation  $t_i^{\mathbf{d}} = 1$  to the above presentation of  $\mathcal{F}_n$ .

**9.3.** As for the Hecke algebra, we can construct an inductive basis for the Y–H algebra. To do that, we define in  $Y_{d,n}$ , the following sets:

$$\mathcal{R}_0 = \{1, t_1^a; 1 \leq a \leq d-1\}, \quad \mathcal{R}_1 = \{1, t_2^a, g_1 r; r \in \mathcal{R}_0, 1 \leq a \leq d-1\}$$

and

$$\mathcal{R}_k = \{1, t_{k+1}^a, g_k r; r \in \mathcal{R}_{k-1}, 1 \leq a \leq d-1\}, \quad \text{for } 2 \leq k \leq n-1.$$

**Definition 17.** *Every element in  $Y_{d,n}$  of the form  $r_1 \cdots r_{n-1}$ , with  $r_i \in \mathcal{R}_i$ ,  $0 \leq i \leq n-1$  is called normal word. We denote by  $\mathcal{R}_n$  the set of normal words.*

**Theorem 16.** *The set  $\mathcal{R}_n$  is linear basis for  $Y_{d,n}$ ; hence  $Y_{d,n}$  has dimension  $d^n n!$ .*

Observe that every element of  $\mathcal{R}_n$  has one of the following forms:

$$\mathfrak{r} t_n^a \quad \text{or} \quad \mathfrak{r} g_{n-1} \cdots g_1 t_1^a, \quad (39) \quad \boxed{\mathcal{R}_n}$$

where  $\mathfrak{r} \in \mathcal{R}_{n-1}$  and  $0 \leq a \leq d-1$ .

**Example 13.** *For  $n = 3$ , we have:*

$$\mathcal{R}_0 = \{1, t_1^a; 1 \leq a \leq d-1\}, \quad \mathcal{R}_1 = \{1, t_2^a, g_1 r; r \in \mathcal{R}_0, 1 \leq a \leq d-1\}$$

and

$$\mathcal{R}_2 = \{1, t_3^a, g_2 r; r \in \mathcal{R}_1, 1 \leq a \leq d-1\}$$

*Thus,  $\mathcal{R}_1$  of  $Y_{d,1}$  is  $\mathcal{R}_0$ . The basis  $\mathcal{R}_2$  of  $Y_{d,2}$  is formed by the elements in the form:*

$$t_1^a t_2^b, \quad t_1^a g_1 t_1^b.$$

*The elements of the basis  $\mathcal{R}_3$  of  $Y_{d,3}$ , are of the form:*

$$t_1^a \underline{t_2^b t_3^c}, \quad t_1^a \underline{t_2^b g_2 t_2^c}, \quad t_1^a \underline{t_2^b g_2 g_1 t_1^c}, \quad t_1^a g_1 \underline{t_1^b t_3^c}, \quad t_1^a g_1 \underline{t_1^b g_2 t_2^c}, \quad t_1^a g_1 \underline{t_1^b g_2 g_1 t_1^c}.$$

*We used the underline to indicate the form of (39) for the elements of  $\mathcal{R}_3$ .*

Now, in particular, (39) said that  $\mathcal{R}_{n-1} \subset \mathcal{R}_n$ , for all  $n$ . Then, for every  $d$  we have the following tower of algebras:

$$Y_{d,1} \subset Y_{d,2} \cdots \subset \cdots Y_{d,n} \subset Y_{d,n+1} \subset \cdots \quad (40) \quad \boxed{\text{tower}Y_n}$$

**Notation 4.** We denote by  $Y_{d,\infty}$  the algebra associated to the tower of algebras above.

**9.4.** Set  $x_0 := 1$  and let  $x_1, \dots, x_{d-1}$  be  $d - 1$  independent parameters commuting among them and with the parameters  $z$ .

**Theorem 17.** The algebra  $Y_{d,\infty}$  supports a unique Markov trace, that is, a family  $\text{tr}_d = \{\text{tr}_{d,n}\}_{n \in \mathbb{N}}$  of linear maps

$$\text{tr}_{d,n} : Y_{d,n} \longrightarrow \mathbb{C}[z, x_1, \dots, x_{d-1}]$$

defined uniquely by the following rules:

- (1)  $\text{tr}_{d,n}(1) = 1$ ,
- (2)  $\text{tr}_{d,n}(\mathbf{ab}) = \text{tr}_{d,n}(\mathbf{ba})$ ,
- (3)  $\text{tr}_{d,n+1}(\mathbf{ag}_n) = z \text{tr}_{d,n}(\mathbf{a})$ ,
- (4)  $\text{tr}_{d,n+1}(\mathbf{at}_{n+1}^k) = x_k \text{tr}_{d,n}(\mathbf{a})$ ,

where  $\mathbf{a}, \mathbf{b} \in Y_{d,n}$ ,  $0 \leq k \leq d - 1$ .

Notice that for  $d = 1$  the trace  $\text{tr}$  becomes the Ocneanu trace.

From now on we fix a positive integer  $d$ , thus we shall write  $\text{tr}_n$  instead of  $\text{tr}_{d,n}$ . Moreover, whenever it is not necessary to explicit  $n$ , we write simply  $\text{tr}$  instead of  $\text{tr}_n$ .

We compute some values of  $\text{tr}$  that will be used later. First, we compute  $\text{tr}(\alpha \mathbf{e}_n \mathbf{g}_n)$ , for every  $\alpha \in Y_{d,n}$ .

Notice that by using the rule of commutation  $\mathbf{t}_n^s \mathbf{t}_{n+1}^{d-s} \mathbf{g}_n = \mathbf{t}_n^s \mathbf{g}_n \mathbf{t}_n^{d-s}$ . Then, for every  $\alpha \in Y_{d,n}$ , we have:

$$\text{tr}(\alpha \mathbf{t}_n^s \mathbf{t}_{n+1}^{d-s} \mathbf{g}_n) = \text{tr}(\alpha \mathbf{t}_n^s \mathbf{g}_n \mathbf{t}_n^{d-s}) = z \text{tr}(\alpha \mathbf{t}_n^s \mathbf{t}_n^{d-s}) = \text{tr}(\alpha).$$

Thus  $\text{tr}(\alpha \mathbf{e}_n \mathbf{g}_n) = d^{-1} \sum_s \text{tr}(\alpha \mathbf{t}_n^s \mathbf{t}_{n+1}^{d-s} \mathbf{g}_n) = z d^{-1} \sum_s \text{tr}(\alpha)$ . Then,

$$\text{tr}(\alpha \mathbf{e}_n \mathbf{g}_n) = z \text{tr}(\alpha) = \text{tr}(\alpha) \text{tr}(\mathbf{g}_n). \quad (41) \quad \boxed{\text{treng}}$$

Now, we compute  $\text{tr}(\mathbf{e}_i)$ .

$$\text{tr}(\mathbf{e}_i) = \frac{1}{d} \sum_s \text{tr}(\mathbf{t}_i^s \mathbf{t}_{i+1}^{d-s}) = \frac{1}{d} \sum_s \text{tr}(\mathbf{t}_i^s) \text{tr}(\mathbf{t}_{i+1}^{d-s}) = \frac{1}{d} \sum_s x_s x_{d-s}.$$

Hence for all  $i$  the trace  $\text{tr}$  takes the same values on  $\mathbf{e}_i$ ; we denote these values by  $E$ . More generally, we define the elements  $\mathbf{e}_i^{(m)}$  as follows,

$$\mathbf{e}_i^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} \mathbf{t}_i^{m+s} \mathbf{t}_{i+1}^{d-s}, \quad \text{where } 0 \leq m \leq d-1, \quad (42) \quad \boxed{\text{em}}$$

where the subindices are regarded as module  $d$ . Then, denoting  $\text{tr}(\mathbf{e}_i^{(m)})$  by  $E^{(m)}$ , we have

$$E^{(m)} := \text{tr}(\mathbf{e}_i^{(m)}) = \frac{1}{d} \sum_{s=0}^{d-1} x_{m+s} x_{d-s} \quad \text{where } 0 \leq m \leq d-1. \quad (43) \quad \boxed{\text{em}}$$

Notice that  $E^{(0)} = E := \text{tr}(\mathbf{e}_i)$ .

The following lemmas will be useful in the next section and their proofs are a good example to see how  $\text{tr}$  works.

trasis1

**Lemma 6.** *Set  $\alpha = w \mathbf{t}_n^k$ , with  $w \in Y_{n-1}$ . We have*

$$\text{tr}(\alpha \mathbf{e}_n^{(m)}) = \frac{E^{(m+k)}}{x_k} \text{tr}(\alpha).$$

*Proof.* A direct computation shows:

$$\text{tr}(w \mathbf{t}_n^k \mathbf{t}_n^{m+s} \mathbf{t}_{n+1}^{d-s}) = x_{d-s} \text{tr}(w \mathbf{t}_n^{m+k+s}) = x_{d-s} x_{m+k+s} \text{tr}(w).$$

Then

$$\text{tr}(\alpha \mathbf{e}_n^{(m)}) = \frac{1}{d} \sum_s \text{tr}(w \mathbf{t}_n^k \mathbf{t}_n^{m+s} \mathbf{t}_{n+1}^{d-s}) = \frac{1}{d} \sum_s x_{d-s} x_{m+k+s} \text{tr}(w),$$

so,  $\text{tr}(\alpha \mathbf{e}_n^{(m)}) = \text{tr}(w) E^{(m+k)}$ . Now,  $\text{tr}(\alpha) = x_k \text{tr}(w)$ . Therefore, the proof follows.  $\square$

trbasis2

**Lemma 7.** *Set  $\alpha = w g_{n-1} g_{n-2} \cdots g_i \mathbf{t}_i^k$ , with  $w \in Y_{n-1}$ . We have*

$$\text{tr}(\alpha \mathbf{e}_n) = z \text{tr}(\alpha' \mathbf{e}_{n-1}),$$

where  $\alpha' = g_{n-2} \cdots g_i \mathbf{t}_i^k w$ .

*Proof.* We have to compute firstly the trace of  $A_s$ ,

$$A_s := w g_{n-1} g_{n-2} \cdots g_i t_i^k t_n^s t_{n+1}^{d-s}.$$

We have

$$\text{tr}(A_s) = x_{d-s} \text{tr}(w g_{n-1} g_{n-2} \cdots g_i t_i^k t_n^s) = x_{d-s} \text{tr}(w t_{n-1}^s g_{n-1} g_{n-2} \cdots g_i t_i^k),$$

where the second equality is obtained by moving  $t_{n-1}^s$  to the left. Now, by using the trace rules (2) and (4), we get

$$\text{tr}(A_s) = z x_{d-s} \text{tr}(w t_{n-1}^s g_{n-2} \cdots g_i t_i^k) = z x_{d-s} \text{tr}((g_{n-2} \cdots g_i t_i^k)(w t_{n-1}^s)),$$

By using again the trace rule (4), we obtain

$$\text{tr}(A_s) = z \text{tr}(\alpha' t_{n-1}^s t_n^{d-s}).$$

Then,

$$\text{tr}(\alpha e_n) = \frac{1}{d} \sum_s \text{tr}(A_s) = \frac{1}{d} \sum_s z \text{tr}(\alpha' t_{n-1}^s t_n^{d-s}) = z \text{tr}(\alpha' e_{n-1}).$$

□

## 10. THE E-SYSTEM

The E-system is the following non-linear system equation of  $(d-1)$  equation in the variable  $x_1, \dots, x_{d-1}$ .

$$\begin{aligned} E^{(1)} &= x_1 E \\ E^{(2)} &= x_2 E \\ &\vdots \\ E^{(d-1)} &= x_{d-1} E. \end{aligned}$$

**Example 14.** For  $d = 3$ , the E-system is:

$$\begin{aligned} x_1 + x_2^2 &= 2x_1^2 x_2 \\ x_1^2 + x_2 &= 2x_1 x_2^2. \end{aligned}$$

For  $d = 4$ , the E-system is:

$$\begin{aligned} x_1 + 2x_2x_3 &= 2x_1^2x_3 + x_1x_2^2 \\ x_1^2 + x_2 + x_3^2 &= 2x_1x_2x_3 + x_2^3 \\ x_3 + 2x_1x_2 &= 2x_1x_3^2 + x_2^2x_3. \end{aligned}$$

The E-system plays a key role to define certain invariants in the next section. The E-system was solved by P. Gerardin, see [27, Appendix].

**Theorem 18** (P. Gerardin, 2013). *The solutions of the E-system are parametrized by the non-empty subset of the group  $\mathbb{Z}/d\mathbb{Z}$ . Moreover, given a such subset  $S$ , the solutions are:*

$$x_k = \frac{1}{|S|} \sum_{s \in S} \exp\left(\frac{2\pi ks}{d}\right), \quad 0 \leq k \leq d-1.$$

**Remark 11.** Let  $S$  be a non-empty subset of  $\mathbb{Z}/d\mathbb{Z}$ . We have:

- (1) For  $|S| = 1$ , the solutions of the E-system are the  $d$ -roots of the unity.
- (2) If  $S$  consists of the coprimes with  $d$ , the solution of the E-system are Ramanujan sums.
- (3) If  $x_1, \dots, x_{d-1}$  is the solution of the E-system parametrized by  $S$ , then

$$E = \text{tr}(e_i) = \frac{1}{|S|}. \quad (44) \quad \boxed{\text{tr}(e_i)}$$

$\boxed{\text{tralphaen}}$

**Proposition 13.** *If the parameters trace  $x_k$ 's are taken as solution of the E-system, then for every  $\alpha \in Y_{d,n}$ , we have:*

$$\text{tr}(\alpha e_n) = \text{tr}(\alpha) \text{tr}(e_n).$$

*Proof.* From the linearity of  $\text{tr}$ , it is enough to consider  $\alpha$  in the basis  $\mathcal{R}_n$ . The proof will be done by induction on  $n$ . For  $n = 1$ , the claim is clear since  $\alpha \in Y_1 = \mathbb{K}$ ; suppose now the proposition be true for every  $n - 1$  and let  $\alpha \in \mathcal{R}_{n-1}$ . Because (39), we have two cases according to the form of  $\alpha$ : (i)  $\alpha = \mathbf{r} \mathbf{t}_n^\alpha$  or (ii)  $\alpha = \mathbf{r} \mathbf{g}_{n-1} \dots \mathbf{g}_1 \mathbf{t}_n^\alpha$ , where  $\mathbf{r} \in \mathcal{R}_{n-1}$  and  $0 \leq \alpha \leq d - 1$ .

For the case (i), from Lemma 6 we have:  $\text{tr}(\alpha e_n) = (E^k/x_k) \text{tr}(\alpha) = E \text{tr}(\alpha)$  since the  $x_k$ 's are solution of the E-system.

For the case (ii), we use the Lemma 7, so:  $\text{tr}(\alpha e_n) = z \text{tr}(\alpha' e_{n-1})$ , where  $\alpha' = g_{n-2} \cdots g_i t_i^k r$ . From the inductive hypotesis we have  $\text{tr}(\alpha' e_{n-1}) = \text{tr}(\alpha') \text{tr}(e_{n-1})$ ; but  $z \text{tr}(\alpha') = \text{tr}(\alpha)$ . Then,  $\text{tr}(\alpha e_n) = \text{tr}(\alpha) \text{tr}(e_{n-1}) = \text{tr}(\alpha) \text{tr}(e_n)$ .  $\square$

### 11. THE INVARIANTS $\Delta_m$ AND $\Theta_m$

In this section we define the invariants  $\Delta_m$  and  $\Theta_m$  which are constructed using the method due to V. Jones to construct the Homflypt polynomial, see Section 6. Essentially, in the Jones method we use now the Y-H algebra instead of the Hecke algebra and instead of the Ocneanu trace we use the trace  $\text{tr}$ . Observe that the construction below follows what has been done in Section 6.

**11.1.** For  $\theta$ , we denote by  $\pi_\theta$  the homomorphism from  $B_n$  to  $Y_{d,n}$ , such that  $\sigma_i \mapsto \theta g_i$ . Since  $\pi_\theta$  is an homomorphism and the trace rule (2) of  $\text{tr}$ , it follows that  $\text{tr} \circ \pi_\theta$  agrees with the Markov replacement M1; thus, it remains to see if  $\text{tr} \circ \pi_\theta$  agrees with the Markov replacements M2, that is, we need

$$(\text{tr} \circ \pi_\theta)(\sigma \sigma_n^{-1}) = (\text{tr} \circ \pi_\theta)(\sigma \sigma_n) \quad \text{for all } \sigma \in B_n. \quad (45) \quad \text{theta1}$$

Put  $\alpha := \pi_\theta(\sigma)$ , then we have  $(\text{tr} \circ \pi_\theta)(\sigma \sigma_n) = \theta \text{tr}(\alpha g_n) = z \theta \text{tr}(\alpha)$  and  $(\text{tr} \circ \pi_\theta)(\alpha \sigma_n^{-1}) = \theta^{-1} \text{tr}(\alpha g_n^{-1})$ . So, the equation (45) is equivalent to:

$$\theta^2 = \frac{\text{tr}(\alpha g_n^{-1})}{z \text{tr}(\alpha)}. \quad (46) \quad \text{theta2}$$

Now, applying the formula (36) to  $g_n^{-1}$  and the linearity of  $\text{tr}$ , we get:

$$\text{tr}(\alpha g_n^{-1}) = \text{tr}(\alpha g_n) + (u^{-1} - 1) \text{tr}(\alpha e_n) + (u^{-1} - 1) \text{tr}(\alpha e_n g_n). \quad (47) \quad \text{theta3}$$

Therefore, in order to get the condition on  $\theta$  in (46), we need to factorize by  $\text{tr}(\alpha)$  in the second member of the equality of (47). Now, observe that  $\text{tr}(\alpha g_n) = z \text{tr}(\alpha)$  and from (41) we get  $\text{tr}(\alpha e_n g_n) = z \text{tr}(\alpha)$ . Unfortunately, we cannot take out the factor  $\text{tr}(\alpha)$  in  $\text{tr}(\alpha e_n)$ . However,

resorting to Proposition 13, we can do it whenever the  $x_i$ 's are the solution of the  $E$ -system. Thus, if  $x_S := (x_1, \dots, x_{d-1})$  is the solution of the  $E$ -system determined by the set  $S$ , the equation (47) can be written as

$$\mathrm{tr}(\alpha g_n^{-1}) = \mathrm{tr}(\alpha) \mathrm{tr}(g_n) + (u^{-1} - 1) \mathrm{tr}(\alpha) \mathrm{tr}(e_n) + (u^{-1} - 1) \mathrm{tr}(\alpha) \mathrm{tr}(e_n g_n).$$

Then, (46) yields

$$\theta^2 = \frac{z + (u^{-1} - 1)E + (u^{-1} - 1)z}{z} = \frac{(1 - u)E + z}{uz}.$$

So, by (44) we get:

$$\theta^2 = \frac{(1 - u) + z|S|}{uz|S|}.$$

Because  $\theta^2$  is depending only on the cardinal of  $S$ , we can define  $\lambda_m$  for every  $m \in \mathbb{N}$ , as the  $\theta^2$  above, i.e.

$$\lambda_m := \frac{(1 - u) + zm}{uzm}. \quad (48)$$

lambda

Recapitulating, by extending the field  $\mathbb{K}$  to  $\mathbb{K}(\sqrt{\lambda_m})$ , we can consider the following homomorphism  $\pi_{\sqrt{\lambda_m}}$ ,

$$\pi_{\sqrt{\lambda_m}} : B_n \longrightarrow Y_{d,n}, \quad \sigma_i \mapsto \sqrt{\lambda_m} g_i.$$

Thus, we have a family of functions  $\{\mathrm{tr}_n \circ \pi_{\sqrt{\lambda_m}}\}_n$  agreeing with the replacements M1 and M2. Proceeding as in (16) we get that, now, the factor of normalization for  $\mathrm{tr}_n \circ \pi_{\sqrt{\lambda_m}}$  is  $(z\sqrt{\lambda_m})^{n-1}$ .

**Definition 18.** Let  $m \in \mathbb{N}$ . For  $\sigma \in B_n$ , we define

$$\Delta_m(\sigma) := \left( \frac{1}{z\sqrt{\lambda_m}} \right)^{n-1} (\mathrm{tr}_n \circ \pi_{\sqrt{\lambda_m}})(\sigma).$$

**Theorem 19.** Let  $L$  be a link s.t.  $L = \widehat{\sigma}$ , where  $\sigma \in B_n$ , then the map  $\Delta_m$  is an invariant of ambient isotopy for oriented links,

$$\Delta_m : \mathcal{L} \longrightarrow \mathbb{C}(z, \sqrt{\lambda_m}), \quad L \mapsto \Delta_m(\sigma).$$

Delta1

**Remark 12.** Regarding (14) and (48) it results clear that  $\Delta_1$  is the Homflypt polynomial.

For the benefit of the writing, the main results on the invariant  $\Delta_m$  will be established after we define a cousin  $\Theta_m$  of  $\Delta_m$ . These invariants are not equivalents but share several properties.

$Y_n(v)$

**11.2.** The Yokonuma–Hecke algebra has another presentation due to M. Chlouveraki and L. Poulain d’Andecy [13], cf. [14, 18]. This presentation is constructed as follows. Firstly, the field  $\mathbb{K}$  is extended to  $\mathbb{K}(v)$  with  $v^2 = u$ ; secondly, new generators  $f_i$  are defined by

$$f_i := g_i + (v^{-1} - 1)e_i g_i, \quad (1 \leq i \leq n - 1). \quad (49) \quad \text{deffi}$$

It is a routine to check that the  $f_i$ ’s and the  $e_i$ ’s satisfy the relations (30)–(34) if one substitutes  $g_i$  with  $f_i$  and the relation (35) becomes

$$f_i^2 = 1 + (v - v^{-1})e_i f_i. \quad (50) \quad \text{quadfi}$$

Notice that  $f_i$ ’s are invertibles, and

$$f_i^{-1} = f_i - (v - v^{-1})e_i. \quad (51) \quad \text{inversefi}$$

Thus one obtains a presentation of the Yokonuma–Hecke algebra by the generators  $f_i$ ’s and  $e_i$ ’s and the same relations of those given for the defining generators  $g_i$ ’s and  $e_i$ ’s except for the relation (35) which is replaced by the relation (50). We shall use the notation  $Y_{d,n}(v)$ , or simply  $Y_n(v)$ , whenever the Yokonuma–Hecke algebra is considered with the presentation by the  $f_i$ ’s and  $e_i$ ’s.

**Remark 13.** Notice that  $Y_{1,n}(v)$  coincides with the presentation of the Hecke through the generators  $\tilde{h}_i$ ’s of Exercise X.

By using now, in the Jones recipe,  $Y_{d,n}(v)$  instead of  $Y_{d,n}(u)$ , one obtains an invariant  $\Theta_m$  instead  $\Delta_m$ . To be precise, notice firstly that for every  $\alpha \in Y_{d,n}$ , according to properties of  $\text{tr}$ , (49) and (41), we have:

$$\text{tr}(\alpha f_n) = \text{tr}(g_n)\text{tr}(\alpha) + (v^{-1} - 1)\text{tr}(g_n)\text{tr}(e_n \alpha) = z v^{-1}\text{tr}(\alpha).$$

Secondly, given a non-empty subset  $\mathcal{S}$  of  $\mathbb{Z}/d\mathbb{Z}$  and  $(x_1, \dots, x_{d-1})$  the solutions determined by  $\mathcal{S}$ , for every  $\alpha \in Y_{d,n}$  the Proposition 13 together with (49) and (51,) imply:

$$\begin{aligned} \text{tr}(\alpha f_n^{-1}) &= v^{-1} \text{tr}(\mathfrak{g}_n) \text{tr}(\alpha) - (v - v^{-1}) \text{tr}(\alpha) \text{tr}(e_n) \\ &= \text{tr}(\alpha) (v^{-1}z - (v - v^{-1})E). \end{aligned}$$

Therefore, the rescaling factor is:

$$\frac{\text{tr}(\alpha f_n^{-1})}{\text{tr}(\alpha f_n)} = \frac{z v^{-1} - (v - v^{-1})E}{z v^{-1}} = \frac{z - (v^2 - 1)E}{z} = \frac{|\mathcal{S}|z - (v^2 - 1)}{|\mathcal{S}|z},$$

the last equality is due to (44). Again the rescaling factor depends only on the cardinal  $|\mathcal{S}|$ , so, by using the procedure used to get the definition of  $\Delta_m$ , we get firstly the rescaling factor, denoted by  $\lambda'_m$ , namely

$$\lambda'_m = \frac{z m - (v^2 - 1)}{z m} \quad (m \in \mathbb{N}); \quad (52)$$

lambdamprime

and secondly  $\Theta_m$  as follows.

Theta

**Definition 19.** For  $\sigma \in B_n$ , we define

$$\Theta_m(\sigma) := \left( \frac{v}{z \sqrt{\lambda'_m}} \right)^{n-1} (\text{tr}_n \circ \pi_{\sqrt{\lambda'_m}})(\sigma), \quad (m \in \mathbb{N}).$$

Keeping the notations above, we have the following theorem.

**Theorem 20.** The function  $\Theta_m : \mathcal{L} \rightarrow \mathbb{K}(z, \sqrt{\lambda'_m})$ , defined by  $L \mapsto \Theta_m(\sigma)$ , where  $L$  is the closure of the braid  $\sigma$ , is an invariant of ambient isotopy of oriented links.

**Remark 14.** Exactly as  $\Delta_1$  in Remark 12, we have that  $\Theta_1$  coincide with the Homflypt polynomial.

**Theorem 21.** The invariants  $\Delta_m$  and  $\Theta_m$  coincide with the Homflypt polynomial whenever they are evaluated on knots.

To prove that the invariants  $\Delta_m$  and  $\Theta_m$  were not equivalent to the Homflypt polynomial was a not trivial matter. Firstly, in [14] were found six pairs of links with equivalent Homflypt polynomial but different  $\Theta_m$ , see [14, Table1]; a such pair is shown in Fig. 20.

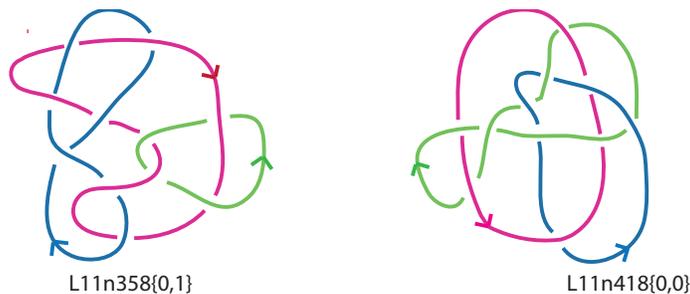


FIGURE 20

FirstTheta

Hence.

**Theorem 22.** *For every  $m \geq 2$ , the invariants  $\Theta_m$  are not equivalent to the Homflypt polynomial.*

The people working on these invariants thought that the invariants  $\Theta_m$  and  $\Delta_m$  were equivalent<sup>1</sup> but surprisingly F. Aicardi shows that it is not the case.

**Theorem 23** (Aicardi). *For every  $m \geq 2$ , the invariants  $\Delta_m$  and  $\Theta_m$  are not topologically equivalent to the Homflypt polynomial: indeed, there exists pairs of non isotopic links distinguished by  $\Delta_m$  and/or  $\Theta_m$  but not by the Homflypt polynomial.*

*Proof.* In [7] we can find several pairs proving this theorem. In Fig. 21, we have a pair of non isotopic links distinguished by  $\Delta_m$ ,  $m \geq 2$ , but neither by Homflypt nor  $\Theta_m$ .

<sup>1</sup>This is due to the fact that in the definitions of  $\Theta_m$  and  $\Delta_m$  the unique difference is the change of presentations for the Y–H algebra, apparently unimportant thing.

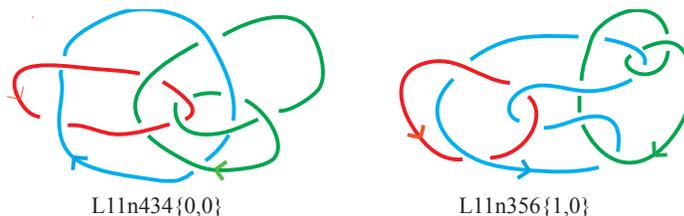


FIGURE 21

DistingueDeltaNoTheta

□

The next theorems establish the main properties of the invariants  $\Delta_m$  and  $\Theta_m$ .

**Proposition 14.** *The invariants  $\Delta_m$  and  $\Theta_m$  share several properties with the Homflypt polynomial, e.g.: the behavior under connected sums and mirror image.*

In the next section we generalize, respectively, the invariants  $\Delta_m$  and  $\Theta_m$  to, respectively, certain invariants in three parameters,  $\bar{\Delta}$  and  $\bar{\Theta}$  for classical links. We will do this, by using the Jones recipe applied to the bt–algebra. Also we will see, that  $\bar{\Theta}$  can be defined through skein relations, while  $\bar{\Delta}$  cannot.

## 12. THE BT–ALGEBRA

In this section we introduce the so–called bt–algebra (or algebra of braids and ties) which is constructed by abstracting the braid generators  $g_i$ ’s of the Yokonuma–Hecke algebra and the idempotents  $e_i$ ’s appearing in the square of the braid generators, see (35). This algebra is used to generalize the invariants  $\Delta_m$  and  $\Theta_m$  to invariants with three parameters as well as its understanding by skein relations. Before introducing the bt–algebra we shall recall the main facts on set partitions since these facts will be useful in the rest of these notes.

**12.1.** For  $n \in \mathbb{N}$ , we denote by the set  $\{1, \dots, n\}$  and by  $P_n$  the set formed by the set partitions of  $[n]$ , that is, an element of  $P_n$  is a collection  $I = \{I_1, \dots, I_k\}$  of pairwise-disjoint non-empty sets whose union is  $[n]$ ; the sets  $I_1, \dots, I_k$  are called the blocks of  $I$ ; the cardinal of  $P_n$ , denoted  $b_n$ , is called the  $n^{\text{th}}$  Bell number.

We can regard  $P_n$  as subset of  $P_{n+1}$  through the natural injective map  $\iota_n : P_n \rightarrow P_{n+1}$ , where for  $I \in P_n$ , the image  $\iota_n(I) \in P_{n+1}$  is defined by adding to  $I$  the block  $\{n + 1\}$ .

Typically, the set partitions are represented by scheme of arcs, see [31, Subsection 3.2.4.3], that is: the point  $i$  is connected by an arc to the point  $j$ , if  $j$  is the minimum in the same block of  $i$  satisfying  $j > i$ . Figure 22 shows the set partition  $I = \{\{1, 3\}, \{2, 5, 6\}, \{4\}\}$  as represented by arcs.

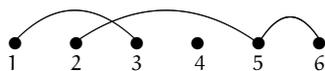


FIGURE 22

SetPartition

The representation by arcs of a set partition induces a natural indexation of its blocks. More precisely, we say that the blocks  $I_j$ 's of the set partition  $I = \{I_1, \dots, I_m\}$  of  $[n]$  are *standard indexed* if  $\min(I_j) < \min(I_{j+1})$ , for all  $j$ . For instance, in the set partition of Figure 22 the blocks are indexed as:  $I_1 = \{1, 3\}$ ,  $I_2 = \{2, 5, 6\}$  and  $I_3 = \{4\}$ .

For our proposal it is convenient to represent a set partitions by *ties*. A representation by ties consists in to replace

The natural action of  $S_n$  on  $[n]$  induces, in the obvious way, an action of  $S_n$  on  $P_n$  that is, for  $I = \{I_1, \dots, I_m\}$  we have

$$w(I) := \{w(I_1), \dots, w(I_m)\}. \tag{53} \quad \text{SnOnPn}$$

Notice that this action preserves the cardinal of each block of the set partition.

Now, we shall say that two set partitions  $I$  and  $I'$  in  $\mathcal{P}_n$  are conjugate, denoted by  $I \sim I'$ , if there exists  $w \in \mathcal{S}_n$  such that,  $I' = w(I)$ ; if it is necessary to precise such  $w$ , we write  $I \sim_w I'$ . Further, observe that if  $I$  and  $I'$  are standard indexed with  $m$  blocks, then the permutation  $w$  induces a permutation of  $\mathcal{S}_m$  of the indices of the blocks, which we denote by  $w_{I,I'}$ .

exw **Example 15.** Let  $I = \{\{1, 2\}_1, \{3\}_2, \{4, 5\}_3, \{6\}_4\}$  and

$$I' = \{\{1\}_1, \{2, 5\}_2, \{3, 6\}_3, \{4\}_4\},$$

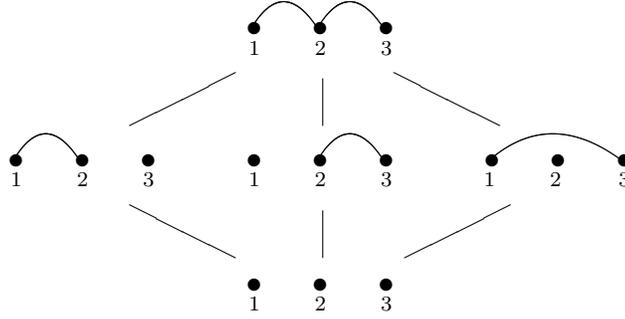
so  $n = 6$  and  $m = 4$ . We have  $I \sim_w I'$ , where:

$$w = (1, 6)(2, 3, 4, 5) \quad \text{and} \quad w_{I,I'} = (1, 3, 2, 4).$$

Given a permutation  $w \in \mathcal{S}_n$  and writing  $w = c_1 \cdots c_m$  as product of disjoint cycles, we denote by  $I_w$  the set partition whose blocks are the cycles  $c_i$ 's, regarded now as subsets of  $\mathbf{n}$ . Reciprocally, given a set partition  $I = \{I_1, \dots, I_m\}$  of  $\mathbf{n}$  we denote by  $w_I$  an element of  $\mathcal{S}_n$  whose cycles are the blocks  $I_i$ 's. Moreover, we shall say that the cycles of  $w_I$  are standard indexed, if they are indexed according to the standard indexation of  $I$ .

**Notation 5.** When there is no risk of confusion, we will omit in the partitions the blocks with a single element.

$\mathcal{P}_n$  is a poset with structure of commutative monoid. Indeed, the partial order on  $\mathcal{P}_n$  is defined as follows:  $I \preceq J$  if and only if each block of  $J$  is a union of blocks of  $I$ . Fig 23 shows the Hasse diagram for  $(\mathcal{P}_3, \preceq)$ .



Hasse

FIGURE 23

The product  $I * J$ , between  $I$  and  $J$  is defined as the minimal set partition, containing  $I$  and  $J$ , according to  $\preceq$ . E.g.

$$\{\{1\}, \{2, 4\}, \{3\}, \{5\}\} * \{\{1, 2\}, \{3\}, \{4, 5\}\} = \{\{1, 2, 4, 5\}, \{3\}\}.$$

Note that  $(P_n, *)$  results a monoid with identity  $\mathbf{1}_n := \{\{1\}, \{2\}, \dots, \{n\}\}$  and that:

$$I * J = J, \quad \text{whenever } I \preceq J, \tag{54} \quad \text{partial}$$

$$I * J = I * w_I(J). \tag{55} \quad \text{astwithw}$$

Further, the injective maps  $\iota_n$ 's result to be a monoid homomorphisms and respect  $\preceq$ , that is, for every  $I, J \in P_n$ , we have:

$$I \preceq J \quad \text{then} \quad \iota_n(I) \preceq \iota_n(J). \tag{56} \quad \text{agreewithposet}$$

**Notation 6.** *The inductive limit associated to the family of monoids  $\{(P_n, \iota_n)\}_{n \in \mathbb{N}}$  is denoted by  $P_\infty$ ,*

We are going to give an abstract description of  $P_n$ , that is, via a presentation which will be used in the next section. For every  $1 \leq i < j \leq n$ , define  $\mu_{i,j} \in P_n$  as the set partition whose blocks are  $\{i, j\}$  and  $\{k\}$  where  $1 \leq k \leq n$  and  $k \neq i, j$ . We shall write  $\mu_{i,j}\mu_{k,h}$  instead of  $\mu_{i,j} * \mu_{k,h}$ .

presentationPn

relationsPn

**Theorem 24** ([19, Theorem 2]). *The monoid  $P_n$  can be presented by the set partitions  $\mu_{i,j}$ 's subject to the following relations:*

$$(1) \mu_{i,j}^2 = \mu_{i,j},$$

- (2)  $\mu_{i,j}\mu_{r,s} = \mu_{r,s}\mu_{i,j}$ ,  
(3)  $\mu_{i,j}\mu_{j,k} = \mu_{i,k}\mu_{i,j} = \mu_{i,k}\mu_{j,k}$ .

**12.2.** The original definition of the bt–algebra is the following.

**Definition 20** (See [2, 4, 33]). *The bt–algebra , denoted by  $\mathcal{E}_n(u)$ , is defined by  $\mathcal{E}_1(u) := \mathbb{K}$  and for  $n \geq 2$  as the unital associative  $\mathbb{K}$ –algebra, with unity 1, defined by braid generators  $T_1, \dots, T_{n-1}$  and ties generators  $E_1, \dots, E_{n-1}$  subjected to the following relations:*

$$E_i E_j = E_j E_i \quad \text{for all } i, j, \quad (57)$$

$$E_i^2 = E_i \quad \text{for all } i, \quad (58)$$

$$E_i T_j = T_j E_i \quad \text{for } |i - j| > 1, \quad (59)$$

$$E_i T_i = T_i E_i, \quad (60)$$

$$E_i T_j T_i = T_j T_i E_i, \quad \text{for } |i - j| = 1 \quad (61)$$

$$E_i E_j T_i = E_j T_i E_j = T_i E_i E_j \quad \text{for } |i - j| = 1, \quad (62)$$

$$T_i T_j = T_j T_i \quad \text{for } |i - j| > 1, \quad (63)$$

$$T_i T_j T_i = T_j T_i T_j \quad \text{for } |i - j| = 1, \quad (64)$$

$$T_i^2 = 1 + (u - 1)E_i + (u - 1)E_i T_i \quad \text{for all } i. \quad (65)$$

The  $T_i$ 's are invertible, with  $T_i^{-1}$  given by:

$$T_i^{-1} = T_i + (u^{-1} - 1)E_i + (u^{-1} - 1)E_i T_i. \quad (66)$$

Tinverse

remarkbt

- Remark 15.** (1) The bt–algebra can be seen as a generalization of the Hecke algebra since by making  $E_i = 1$ , the definition of the bt–algebra becomes the Hecke algebra. Further, observe that the mapping  $T_i \mapsto h_i$  and  $E_i \mapsto 1$  defines an epimorphism from the bt–algebra to the Hecke algebra.
- (2) The mapping  $T_i \mapsto g_i$  and  $E_i \mapsto e_i$  defines an algebra homomorphism from the bt–algebra to the YH–algebra, which is injective if and only if  $d \geq n$ , see [18], cf. [4, Remark 3].

Diagrammatically the generators  $T_i$ 's can be regarded as usual braids and the generator  $E_i$  as a tie between the  $i$  and  $i + 1$  strands, this tie

doesn't have a topological meaning: it is an auxiliary artefact to reflect the monomial-homogeneous defining relation of the  $bt$ -algebra. Thus, the tie is pictured as a spring or a dashed line between the strands. More precisely, the diagrams for, respectively,  $T_i$  and  $E_i$ , are:



FIGURE 24

TiEi

Here comment!

**12.3.** The  $bt$ -algebra is a finite dimensional algebra. Moreover, there is a basis due to S. Ryom-Hansen [33]. Before expliciting the Ryom-Hansen basis we need to introduce the tools below.

For  $i < j$ , we define  $E_{i,j}$  by

$$E_{i,j} = \begin{cases} E_i & \text{for } j = i + 1, \\ T_i \cdots T_{j-2} E_{j-1} T_{j-2}^{-1} \cdots T_i^{-1} & \text{otherwise.} \end{cases} \tag{67} \quad \boxed{E_{ij}}$$

For any nonempty subset  $J$  of  $\mathbf{n}$  we define  $E_J = 1$  for  $|J| = 1$  and otherwise by

$$E_J := \prod_{(i,j) \in J \times J, i < j} E_{i,j}.$$

Note that  $E_{\{i,j\}} = E_{i,j}$ . For  $I = \{I_1, \dots, I_m\} \in P_n$ , we define  $E_I$  by

$$E_I = \prod_k E_{I_k}. \tag{68} \quad \boxed{E_I}$$

Now, if  $w = s_{i_1} \cdots s_{i_k}$  is a reduced expression of  $w \in S_n$ , then the element  $T_w := T_{i_1} \cdots T_{i_k}$  is well defined. The action of  $S_n$  on  $P_n$  is inherited from the  $E_I$ 's and we have:

$$T_w E_I T_w^{-1} = E_{w(I)} \quad (\text{see [33, Corollary 1]}). \tag{69} \quad \boxed{w(E)}$$

basEn

**Theorem 25** ([33, Corollary 3]). *The set  $\{E_I T_w; w \in \mathcal{S}_n, I \in \mathcal{P}_n\}$  is a  $\mathbb{K}$ -linear basis of  $\mathcal{E}_n(\mathbf{u})$ . Hence the dimension of  $\mathcal{E}_n(\mathbf{u})$  is  $b_n n!$ .*

**Example 16.**

Having in mind (56), the natural inclusion  $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ , for every  $n$ , together with Theorem 25, we deduce the tower of algebras:

$$\mathcal{E}_1(\mathbf{u}) \subset \mathcal{E}_2(\mathbf{u}) \subset \cdots \subset \mathcal{E}_n(\mathbf{u}) \subset \mathcal{E}_{n+1}(\mathbf{u}) \subset \cdots$$

**Notation 7.** Denote by  $\mathcal{E}_\infty(\mathbf{u})$  the inductive limit associated the bt-algebras above.

Vi

**Remark 16** (Cf. Subsection 11.2). Extending the field  $\mathbb{K}$  to  $\mathbb{K}(\mathbf{v})$  with  $\mathbf{v}^2 = \mathbf{u}$ , we can define (cf. [32, Subsection 2.3]):

$$V_i := T_i + (\mathbf{v}^{-1} - 1)E_i T_i. \quad (70) \quad \text{defVi}$$

Then the  $V_i$ 's and the  $E_i$ 's satisfy the relations (59)–(64) and the quadratic relation (65) is transformed in

$$V_i^2 = 1 + (\mathbf{v} - \mathbf{v}^{-1})E_i V_i. \quad (71) \quad \text{newbt9}$$

So,

$$V_i^{-1} = V_i - (\mathbf{v} - \mathbf{v}^{-1})E_i. \quad (72) \quad \text{Ttildeinverse}$$

In [14, 18, 22] this quadratic relation is used to define the bt-algebra. Although at algebraic level these algebras are the same, we will see that they lead to different invariants. Thus, in order to distinguish these two presentations of the bt-algebra, we will write  $\mathcal{E}_n(\mathbf{v})$  when the bt-algebra is defined by using the quadratic relation (71).

**12.4.** In [4] it was proved that the bt-algebra supports a Markov trace, this was proved using the method of relative trace and the Ryom-Hansen basis.

Let  $\mathbf{a}$  and  $\mathbf{b}$  be two variables commutative independent commuting with  $\mathbf{u}$ .

tracerho

**Theorem 26** ([4, Theorem 3]). *There exists a unique Markov trace  $\rho$  on  $\mathcal{E}_\infty(\mathbf{u})$ , i.e., a family  $\rho := \{\rho_n\}_{n \in \mathbb{N}}$ , where  $\rho_n$ 's are linear maps, defined inductively, from  $\mathcal{E}_n(\mathbf{u})$  in  $\mathbb{K}[\mathbf{a}, \mathbf{b}]$  such that  $\rho_n(\mathbf{1}) = \mathbf{1}$  and satisfying, for all  $X, Y \in \mathcal{E}_n(\mathbf{u})$ , the following rules:*

- (1)  $\rho_n(XY) = \rho_n(YX)$ ,
- (2)  $\rho_{n+1}(XT_n) = \rho_{n+1}(XT_nE_n) = \mathbf{a} \rho_n(X)$ ,
- (3)  $\rho_{n+1}(XE_n) = \mathbf{b} \rho_n(X)$ .

With this theorem and because the braid group is represented (naturally) in the bt-algebra we are ready to define an invariant for links. More precisely, denote by  $\pi_{\sqrt{\mathbf{L}}}$  the (natural) representation of  $B_n$  in  $\mathcal{E}_n$ , namely  $\sigma_i \mapsto \sqrt{\mathbf{L}}T_i$ . With the same procedure used to get (14) and (48), we define

$$\mathbf{L} := \frac{\mathbf{a} + (1 - \mathbf{u})\mathbf{b}}{\mathbf{a} \mathbf{u}}. \quad (73) \quad \square$$

barDelta

**Definition 21.** *For  $\sigma \in B_n$ , we define*

$$\bar{\Delta}(\sigma) = \left( \frac{1}{\mathbf{a} \sqrt{\mathbf{L}}} \right)^{n-1} (\rho_n \circ \pi_{\sqrt{\mathbf{L}}})(\sigma).$$

DeltaBarra

**Theorem 27.** *Let  $L$  be a link obtained by closing the braid  $\sigma \in B_n$ . Then the map  $L \mapsto \bar{\Delta}(\sigma)$  defines an ambient isotopy invariant for oriented links, taking values in  $\mathbb{K}(\mathbf{a}, \sqrt{\mathbf{L}})$ .*

*Proof.* We need only to prove that  $\bar{\Delta}$  agrees with the Markov replacements. Because  $\pi_{\sqrt{\mathbf{L}}}$  is an homomorphism and the properties of  $\rho$  imply that  $\bar{\Delta}$  agrees with M1. On the second replacement, we note that it is a routine to check  $\bar{\Delta}(\sigma) = \bar{\Delta}(\sigma\sigma_n)$ . Thus it remains only to check  $\bar{\Delta}(\sigma) = \bar{\Delta}(\sigma\sigma_n^{-1})$ , for every  $\sigma \in B_n$ . Now, put  $\alpha = \pi_{\sqrt{\mathbf{L}}}(\sigma)$ , then:

$$\begin{aligned} (\rho_{n+1} \circ \pi_{\sqrt{\mathbf{L}}})(\sigma\sigma_n^{-1}) &= \sqrt{\mathbf{L}}^{-1} \rho_{n+1}(\alpha T_n^{-1}) \\ &= \sqrt{\mathbf{L}}^{-1} \rho_{n+1}(\alpha(T_n + (\mathbf{u}^{-1} - 1)E_n \\ &\quad + (\mathbf{u}^{-1} - 1)E_n T_n)) \\ &= \sqrt{\mathbf{L}}^{-1} (\mathbf{u}^{-1} \rho_n(\alpha) \mathbf{a} + (\mathbf{u}^{-1} - 1) \rho_n(\alpha) \mathbf{b}) \\ &= \sqrt{\mathbf{L}}^{-1} (\mathbf{u}^{-1} \mathbf{a} + (\mathbf{u}^{-1} - 1) \mathbf{b}) \rho_n(\alpha). \end{aligned}$$

Therefore,  $\overline{\Delta}(\sigma\sigma_n^{-1}) = \overline{\Delta}(\sigma)$ , for all  $\alpha \in B_n$ .  $\square$

Now, in the way we define  $\Theta_m$  we can define  $\overline{\Theta}$ , that is, taking out its definition by using now the presentation of the bt-algebra  $\mathcal{E}_n(v)$  in the Jones recipe. Having in mind what we did above, (52) and Definition 19, we define  $L'$  and  $\overline{\Theta}$  as follows:

$$L' := \frac{a - (v^2 - 1)b}{a}; \quad (74) \quad \text{Lprime}$$

$$\overline{\Theta}(\sigma) := \left( \frac{v}{a\sqrt{L'}} \right)^{n-1} (\rho_n \circ \pi_{\sqrt{L'}})(\sigma), \quad (\sigma \in B_n). \quad (75) \quad \text{ThetaPrime}$$

ThetaBarra

**Theorem 28.** *Let  $L$  be a link obtained by closing the braid  $\alpha \in B_n$ . Then the map  $L \mapsto \overline{\Theta}(\alpha)$  defines an invariant of ambient isotopy for oriented links, which take values in  $\mathbb{K}(a, \sqrt{L'})$ .*

*Proof.* Same proof as Theorem 27.  $\square$

**Remark 17.** The invariant  $\overline{\Delta}$  contains the invariant  $\Delta_m$ , for every  $m \in \mathbb{N}$ ; that is specializing the variable  $b$  to  $1/m$  we get  $\Delta_m$ . With the same specialization, the invariant  $\overline{\Theta}$  contains the invariants  $\Theta_m$ , for every  $m$ .

### 13. TIED LINKS

In this section we introduce the tied links. This class of knotted-like objects contains the classical link, so invariants of tied links yield invariants of classical links. We start the section recalling the definition of tied links and the monoid of braid tied links. Also, the respective Alexander and Markov theorems are exhibited.

**13.1.** Tied links were introduced in [3] and roughly correspond to links whose components may be connected by ties; thus ties are connecting pairs of points of two components or of the same component.

The ties in the picture of the tied links are drawn as springs or dashed lines, to outline that they can be contracted and extended, letting their extremes to slide along the components.

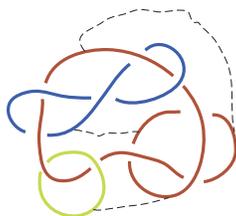


FIGURE 25

ExampleTiedLink

**Notation 8.** We will use the notation  $C_i \rightsquigarrow C_j$  to indicate that either there is a tie between the components  $C_i$  and  $C_j$  of a link, or  $C_i$  and  $C_j$  are the extremes of a chain of  $m > 2$  components  $C_1, \dots, C_m$ , such that there is a tie between  $C_i$  and  $C_{i+1}$ , for  $i = 1, \dots, m - 1$ .

TiedLinks

**Definition 22** ([3, Definition 1.1]). Every 1-link is by definition a tied 1-link. For  $k > 1$ , a tied  $k$ -link is a link whose set of components  $\{C_1, \dots, C_k\}$  is partitioned into parts according to: two components  $C_i$  and  $C_j$  belong to the same part if  $C_i \rightsquigarrow C_{i+1}$ .

**Notation 9.** We denote by  $\tilde{\mathcal{L}}$  the set of oriented tied links.

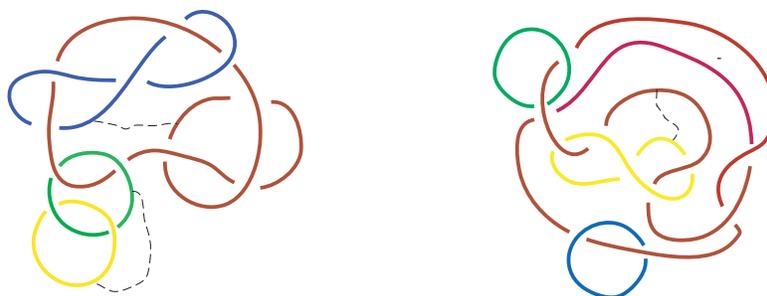


FIGURE 26

ExamplesTiedLinks

In Fig. 26 two tied links with four components; moreover, if  $C_1$  is the blue component,  $C_2$  the red component,  $C_3$  the yellow components and  $C_4$  the green component, then the partition associated to the first tied

link is  $\{\{C_1, C_2\}, \{C_3, C_4\}\}$  and the partitions associated to the second tied link is  $\{\{C_1\}, \{C_2, C_3\}, \{C_4\}\}$ .

Notice that a tied  $k$ -link  $L$ , with components' set  $C_L = \{C_1, \dots, C_k\}$ , determines a pair  $(L, I(C_L))$  in  $\mathfrak{L}_k \times P_k$ , where  $i$  and  $j$  belong to the same block of  $I(C_L)$ , if  $C_i \rightsquigarrow C_j$ .

**Example 17.** For instance in Fig. 26, the set partition determined by the first tied link is  $\{\{1, 2\}, \{3, 4\}\}$  and the set partition determined by the second tied link is  $\{\{1\}, \{4\}, \{2, 3\}\}$ . The set partition determined by the tied link of Fig. 25 is  $\{\{1, 2, 3, 4\}\}$ .

**Definition 23** (Cf. [3, Definition 1.6]). A tie of a tied link is said essential if it cannot be removed without modifying the partition  $I(C_L)$ , otherwise the tie is said unessential.

Notice that between the  $c$  components indexed by the same block of the set partition, the number of essential ties is  $c-1$ ; for instance, in the tied link of Fig. 26, left, among the three ties connecting the first three components, only two are essential. The number of unessential ties is arbitrary. Ties connecting one component with itself are unessential.

tisotopy

**Definition 24.** The  $k$ -tied link  $L$  and the  $k'$ -tied link  $L'$  with, respectively, components  $C = \{C_1, \dots, C_k\}$  and  $C' = \{C'_1, \dots, C'_{k'}\}$ , are  $t$ -isotopic if:

- (1) The links  $L$  and  $L'$  are ambient isotopic (hence  $k = k'$ ).
- (2) The set partitions  $I(C)$  and  $I(C')$  satisfy  $I(C') = w_{L,L'}(I(C))$ , where  $w_{L,L'}$  is the bijection from  $C(L)$  to  $C(L')$  induced by the isotopy.

**Example 18.** In the Fig. 27, we have that  $L_1$  and  $L_2$  are not  $t$ -isotopic. Indeed, the set partition, respectively, of  $L_1$  and  $L_2$  are  $I_1 = \{\{1, 2, 3, 4\}\}$  and  $I_2 = \{\{1, 3, 4\}, \{2\}\}$  and  $w_{L_1, L_2} = (1, 2, 4, 3)$ , but  $I_2 \neq w_{L_1, L_2}(I_1)$ .

Now, L3 has associated the set partition  $I_3 = \{\{1, 2, 3\}, \{4\}\}$  and L4 has associated the set partition  $I_4 = \{\{1\}, \{2, 3, 4\}\}$  and  $w_{L3,L4} = (4, 1, 2, 3)$ ; thus,  $I_4 = w_{L3,L4}(I_3)$ . Then, L3 and L4 are  $t$ -isotopic

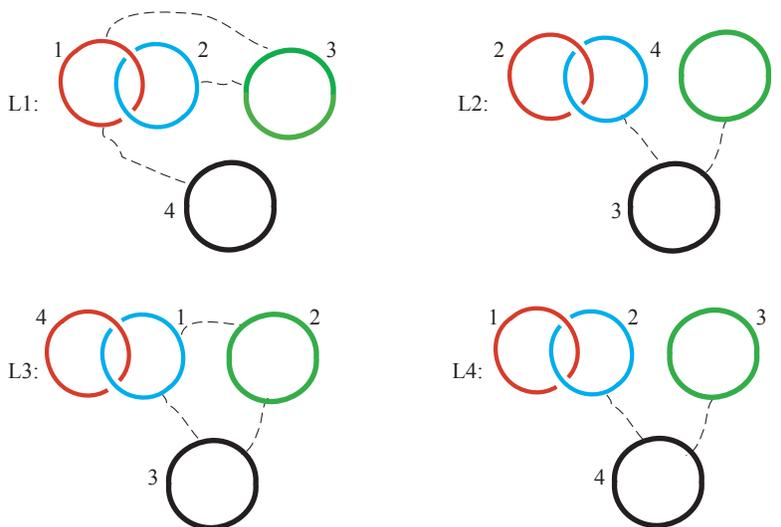


FIGURE 27

```
tisotopyLinks
```

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LinksAsTiedLinks
```

**Remark 18.** The Definitions 22 and 24 not only say that classical links are included in tied links but also that the classical links can be identified to the set of tied links which components are all tied. Both ways to see the classical links allows to study the isotopy of links through the  $t$ -isotopy of tied links. Observe that for  $k$ -tied links with set of components  $C$ , we have  $I(C)$  is  $\{1, 2, \dots, n\}$  if it has all components tied and is  $\{\{1\}, \{2\}, \dots, \{n\}\}$  if does not have ties or has only ties that are unessential.

**Remark 19.** Everything established for tied links can be translated in terms of diagrams in the obvious way. Informally, it is enough to change ‘links’ by ‘diagrams of a link’ and so on.

**13.2.** The classical theorems of Alexander and Markov in knot theory have their analogous in the world of tied links. The starting point

to establish these theorems for tied links is the so-called tied braid monoid.

monoidtb

**Definition 25.** [3, Definition 3.1] *The tied braid monoid  $\mathbb{TB}_n$  is the monoid generated by usual braids  $\sigma_1, \dots, \sigma_{n-1}$  and the tied generators  $\eta_1, \dots, \eta_{n-1}$ , such the  $\sigma_i$ 's satisfy braid relations among them together with the following relations:*

$$\eta_i \eta_j = \eta_j \eta_i \quad \text{for all } i, j \quad (76)$$

$$\eta_i \sigma_i = \sigma_i \eta_i \quad \text{for all } i \quad (77)$$

$$\eta_i \sigma_j = \sigma_j \eta_i \quad \text{for } |i - j| > 1 \quad (78)$$

$$\eta_i \sigma_j \sigma_i = \sigma_j \sigma_i \eta_j \quad \text{for } |i - j| = 1 \quad (79)$$

$$\eta_i \sigma_j \sigma_i^{-1} = \sigma_j \sigma_i^{-1} \eta_j \quad \text{for } |i - j| = 1 \quad (80)$$

$$\eta_i \eta_j \sigma_i = \eta_j \sigma_i \eta_j = \sigma_i \eta_i \eta_j \quad \text{for } |i - j| = 1 \quad (81)$$

$$\eta_i \eta_i = \eta_i \quad \text{for all } i. \quad (82)$$

We denote  $\mathbb{TB}_\infty$  the inductive limit determined by the natural monomorphism monoid from  $\mathbb{TB}_n$  into  $\mathbb{TB}_{n+1}$ .

Diagrammatically, as usual,  $\sigma_i$  is represented as the usual braid and the tied generator  $\eta_i$ , as the diagram of  $E_i$ , that is, a tie connecting the  $i$  with  $(i + 1)$ -strands, see Fig. 24.

The tied braid monoid is to the bt-algebra as the braid group is to the Hecke algebra. In particular, we have the following proposition and its corollary.

TildePi

**Proposition 15.** *The mapping  $\eta_i \mapsto E_i$ ,  $\sigma_i \mapsto T_i$  defines an homomorphism, denoted by  $\tilde{\pi}$ , from  $\mathbb{TB}_n$  to  $\mathcal{E}_n$ .*

*Proof.* □

**Corollary 4.**  $\mathcal{E}_n$  *The bt-algebra is a quotient of  $\mathbb{KTB}_n$ . More precisely,*

$$\mathcal{E}_n \cong \mathbb{KTB}_n / I,$$

where  $I$  is the two-sided ideal generated by  $T_i^2 - (u - 1)E_i(1 + T_i)$ , for  $1 \leq i \leq n - 1$ .

$\mathbb{TB}_n$  has a decomposition like semidirect product of groups, this decomposition allows a study purely algebraic-combinatorics of the tied link, see [6], and will be used below. To establish this decomposition, we start by noting that the action of  $\mathbb{S}_n$  on  $\mathbb{P}_n$ , see (53), together with the natural projection  $\mathfrak{p}$  of (4), define an action of  $\mathbb{B}_n$  on  $\mathbb{P}_n$ . We denote by  $\sigma(I)$ , the action of the braid  $\sigma$  on  $I \in \mathbb{P}_n$ , that is,  $\sigma(I)$  is the result of the application of the permutation  $\mathfrak{p}(\sigma)$  to the set partition  $I$ . Define now the following product in  $\mathbb{P}_n \times \mathbb{B}_n$ :

$$(I, \sigma)(I', \sigma') = (I * \sigma(I'), \sigma\sigma').$$

$\mathbb{P}_n \times \mathbb{B}_n$  with this product is a monoid, which is denoted by  $\mathbb{P}_n \rtimes \mathbb{B}_n$ . We shall denote  $I\sigma$  instead  $(I, \sigma)$ .

Semidirect

**Theorem 29** ([6, Theorem 9]). *The monoid  $\mathbb{TB}_n$  and  $\mathbb{P}_n \rtimes \mathbb{B}_n$  are isomorphic.*

Now, as for braid, we can define the closure of a tied braid in the same way of the closure of the braids. Evidently, the closure of a tied braid is a tied link. Moreover, in [3, Theorem 3.5] we have proved the Alexander theorem for tied links; namely.

**Theorem 30** ([3, Theorem 3.5]). *Every oriented tied link can be obtained as closure of a tied braid.*

Before establishing the Markov theorem for tied links, notice that according to Theorem 29, every element  $\mathfrak{a}$  in  $\mathbb{TB}_n$  can be written uniquely in the form  $\mathfrak{a} = I\sigma$ , where  $I \in \mathbb{P}_n$  and  $\sigma \in \mathbb{B}_n$ . We use this fact in the definition of Markov moves for tied links and also we use the notation of the  $\mu_{i,j}$ 's as in (24).

**Definition 26.** *We say that  $\mathfrak{a}, \mathfrak{b} \in \mathbb{TB}_\infty$  are  $t$ -Markov equivalents, denoted  $\mathfrak{a} \sim_{tM} \mathfrak{b}$ , if  $\mathfrak{b}$  can be obtained from  $\mathfrak{a}$  by using a finite sequence of the following replacements:*

- tM1.  *$t$ -Stabilization: for all  $\mathfrak{a} = I\sigma \in \mathbb{TB}_n$ , we can replace  $\mathfrak{a}$  by  $\mathfrak{a}\mu_{i,j}$ , if  $i, j$  belong to the same cycle of  $\mathfrak{p}(\sigma)$ .*

tM2. *Commuting in  $\mathbb{TB}_n$ . For all  $\mathbf{a}, \mathbf{b} \in \mathbb{TB}_n$ , we can replace  $\mathbf{ab}$  by  $\mathbf{ba}$ ,*

tM3. *Stabilizations: for all  $\mathbf{a} \in \mathbb{TB}_n$ , we can replace  $\mathbf{a}$  by  $\mathbf{a}\sigma_n$  or  $\mathbf{a}\sigma_n^{-1}$ .*

The relation  $\sim_{tM}$  is an equivalence relation on  $\mathbb{TB}_\infty$  and in [3, Theorem 3.7], cf. [6, Theorem 5], the following theorem was proved.

**Theorem 31.** *Two tied braids define  $t$ -isotopic tied links if and only if they are  $t$ -Markov equivalent.*

**Remark 20.** The Markov replacements tM2 and tM3 are the classical Markov moves but including now the ties, so their geometrical meaning is clear. The replacement tM1 says that the ties provided by  $\mu_{i,j}$  in the closure of  $\mathbf{a}\mu_{i,j}$  become unessential ties, so the closure of  $\mathbf{a}$  and  $\mathbf{a}\mu_{i,j}$  defines the same set partition. Below we show an example of how tM1 works.

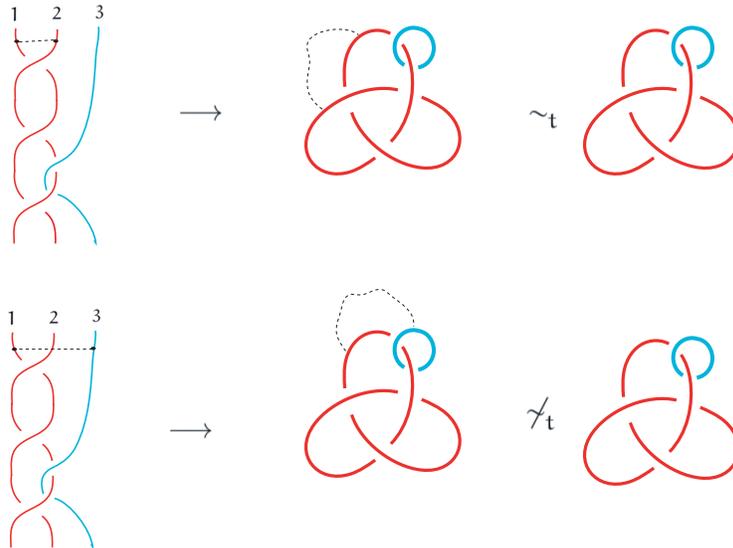


FIGURE 28

□

14. THE INVARIANT  $\mathcal{F}$

We will define the invariant  $\mathcal{F}$  for tied links. This invariant is of type Homflypt since evaluated on classical links coincide with the Homflypt polynomial. We start by defining it by skein relations and later their definition by the Jones recipe.

14.1. To define  $\mathcal{F}$  by skein relations we need, as in (17), to introduce the following notation:  $L_+, L_-, L_0, L_{+,\sim}, L_{-,\sim}$  and  $L_{0,\sim}$  denote oriented tied links which have, respectively, oriented diagrams of tied links  $D_+, D_-, D_0, D_{+,\sim}, D_{-,\sim}$  and  $D_{0,\sim}$ , that are identical outside a small disk into which enter two strands, whereas inside the disk the strands look, respectively, as Fig. 29 shows.

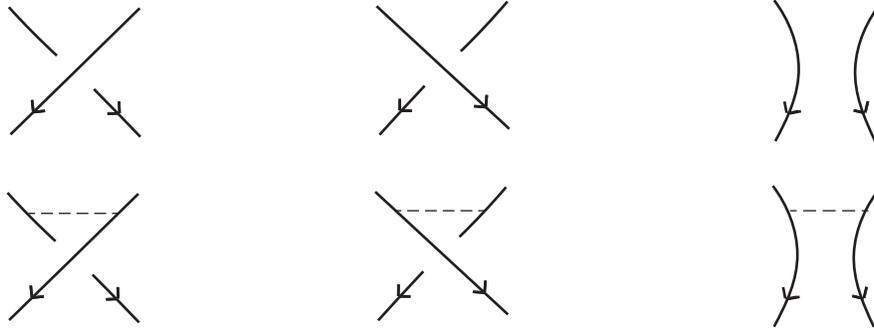


FIGURE 29

ConwayTied

We shall call  $L_{+,\sim}, L_{-,\sim}$  and  $L_{0,\sim}$  a tied Conway triple.

Set  $w$  a variable with unique condition that commutes with  $u$  and  $a$ .

SkeinF

**Theorem 32** ([3, Theorem 2.1]). *There is a unique function  $\mathcal{F} : \tilde{\mathcal{L}} \rightarrow \mathbb{K}(a, w)$ , defined by the rules:*

- (1)  $\mathcal{F}(\bigcirc) = 1$ ,
- (2) For all tied link  $L$ ,

$$\mathcal{F}(\bigcirc \sqcup L) = \frac{1}{aw} \mathcal{F}(L),$$

(3) *The Skein rule:*

$$\frac{1}{w}\mathcal{F}(L_+) - w\mathcal{F}(L_-) = (1 - u^{-1})\mathcal{F}(L_{0,\sim}) - \frac{1 - u^{-1}}{w}\mathcal{F}(L_{+,\sim}).$$

OthersSkeinF

**Proposition 16.** *The defining skein relation of  $\mathcal{F}$  is equivalent to the following skein rules:*

(1)

$$\frac{1}{uw}\mathcal{F}(L_{+,\sim}) - w\mathcal{F}(L_{-,\sim}) = (1 - u^{-1})\mathcal{F}(L_{0,\sim}),$$

(2)

$$\frac{1}{w}\mathcal{F}(L_+) = w[\mathcal{F}(L_-) + (u - 1)\mathcal{F}(L_{-,\sim})] + (u - 1)\mathcal{F}(L_{0,\sim}),$$

(3)

$$w\mathcal{F}(L_-) = \frac{1}{w}[\mathcal{F}(L_+) + (u^{-1} - 1)\mathcal{F}(L_{+,\sim})] + (u^{-1} - 1)\mathcal{F}(L_{0,\sim}).$$

*Proof.*

□

Define  $\tilde{\mathcal{L}}_0$  the set of tied links with all components tied.

FatL0

**Proposition 17.** *The restriction of  $\mathcal{F}$  to  $\tilde{\mathcal{L}}_0$ , is determined uniquely by the rules:*

(1)  $\mathcal{F}(\bigcirc) = 1$ ,

(2) *Skein relation on tied Conway triple  $L_{+,\sim}$ ,  $L_{-,\sim}$  and  $L_{0,\sim}$ ,*

$$r^{-1}\mathcal{F}(L_{+,\sim}) - r\mathcal{F}(L_{-,\sim}) = s\mathcal{F}(L_{0,\sim}),$$

$$\text{where } r = w\sqrt{u} \text{ and } s = \sqrt{u} - \sqrt{u}^{-1}.$$

*Proof.* The computation of  $\mathcal{F}$  on a such tied link can be realized by rules (1) Theorem 32 and the skein relation (1) Proposition 16. Multiplying this skein relations by  $\sqrt{u}$ , we get the values of  $r$  and  $s$ . Hence the proof is concluded. □

**Remark 21.** According to Remark 18,  $\tilde{\mathcal{L}}_0$  is identified to classical links, so, the above proposition and Theorem(12)) say that  $\mathcal{F}$  and Homflypt polynomial coincide on classical links. For this reason we say that  $\mathcal{F}$  is invariant of type Homflypt.

**Proposition 18.** *The restriction of  $\mathcal{F}$  to classical links is more powerful than the Homflypt polynomial.*

*Proof.* □

**14.2.** In order to define  $\mathcal{F}$  through the Jones recipe, we extend first the domain  $\pi_{\sqrt{L}}$  to  $\mathbb{TB}_n$ ; we denote this extension by  $\tilde{\pi}_{\sqrt{L}}$ . Secondly, we define  $\tilde{\Delta}$  by

$$\tilde{\Delta}(\eta) := \left( \frac{1}{a\sqrt{L}} \right)^{n-1} (\rho_n \circ \tilde{\pi}_{\sqrt{L}})(\eta) \quad (\eta \in \mathbb{TB}_n).$$

**Theorem 33.** *Let  $L$  be a tied link obtained as the closure of the tied braid  $\eta$ , then*

$$\tilde{\Delta}(\eta) = \mathcal{F}(L).$$

*The relation among,  $u$ ,  $w$  and parameters trace  $a$  and  $b$  of  $\rho$ , is given by the equation:*

$$b = \frac{a(uw^2 - 1)}{1 - u}.$$

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IMUV, UNIVERSIDAD DE VALPARAÍSO, GRAN BRETAÑA 1111, VALPARAÍSO  
2340000, CHILE.

*Email address:* juyumaya@gmail.com